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SOME COMBINATORIAL PROBLEMS OF ARITHMETIC

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and D. J. NEWMAN, Yeshiva University

We wish to develop techniques for estimating the size of certain collections having given arithmetical properties.

We make no attempt at completeness at the present time but simply present several examples of this type. Future research will aim at a more unified treatment. Our first and simplest problem should serve to indicate the spirit of this investigation, however.

PROBLEM 1. *Determine the maximum number of disjoint couples (of distinct positive integers) whose sums are all distinct and all below n.*

For example let $n=10$. We find the couples (1, 8), (2, 6), (3, 4) with sums, respectively, 9, 8, 7. A bit of experimentation shows that no four couples can be so found and we obtain the answer *three* for the case $n=10$. We will prove that this maximum number is asymptotically $\frac{2}{5}n$. To do so requires two steps, first a direct construction which produces (approximately) $\frac{2}{5}n$ such couples and then a proof that more than this number is impossible.

Step 1. The construction: Let k be the largest positive integer below $n/5$. Form the couples

$$(1, 4k), (3, 4k-1), (5, 4k-2), (7, 4k-3), \cdots, (2k-1, 3k+1)$$

as well as the couples

$$(2, 3k), (4, 3k-1), (6, 3k-2), (8, 3k-3), \cdots, (2k-2, 2k+2).$$

These couples are clearly disjoint and have all sums different. Furthermore the largest sum is $2k-1+3k+1=5k<n$. Thus we have produced an acceptable collection of $2k-1$ couples and $2k-1 \geqq 2\{(n)/5-1)\}-1=\frac{2}{5}n-3$.

Step 2. The proof: Let there be given an acceptable set consisting of N couples. We compute the sum of all the elements in these couples in two different ways. First it is the sum of $2N$ distinct positive integers and as such it is $\geqq 1+2+3+\cdots 2N=N(2N+1)$. Again it is the sum of the N *couple sums* and so is the sum of N distinct integers below n. As such it is $\leqq (n-1)+(n-2)+\cdots(n-N)=(N/2)(2n-N-1)$. Combining these two inequalities, we have $N(2N+1)\leqq(N/2)(2n-N-1)$ or $N\leqq(2n-3)/5$.

Summarizing, we have

THEOREM 1. *The maximum number, N, of disjoint couples of positive integers with distinct sums all below n satisfies*

$$\frac{2n-3}{5} \geqq N \geqq \frac{2n}{5} - 3.$$

(We might remark that for n of the form $5k+1$, N is exactly equal to $2k-1$.)

We turn now to our second problem.

PROBLEM 2. *Determine the maximum number of disjoint triples (of distinct positive integers) having sum equal to n.*

The answer, we will show, is asymptotically $\frac{2}{9}n$.

For this problem the proof portion is the easy half and we give it first. We prove that this maximum number N satisfies $N \leqq (2n-3)/9$. The proof is as before, namely, assume we have a collection of N such triples and compute the overall sum in two different ways. First there are $3N$ distinct positive integers so that the sum is $\geqq 1+2+\cdots+3N=(3/2)N(3N+1)$. Also the sum is exactly $n \cdot N$. Thus,

$$nN \geqq (3/2)N(3N+1) \quad \text{and so} \quad N \leqq (2n-3)/9.$$

Now for the construction. We shall, in fact, add the further restriction that no integer above $\frac{2}{3}n+2$ be used in any of the triples, and still produce "around" $\frac{2}{9}n$ of them. This will be done in parts with the same fundamental step repeated very often.

Let k be the largest integer of the same parity of n which is $\leqq n/6$. Form the triples

$$(1, 3k-1, n-3k), (2, 3k-3, n-3k+1), (3, 3k-5, n-3k+2), \cdots, (k, k+1, n-2k+1)$$

and thus obtain k disjoint triples of integers $\leqq \frac{2}{3}n+2$ each summing to n.

Now note that the numbers $k+2, k+4, \cdots, n-3k-1$ have not been used in the above construction. We can use them to build new triples! For three of these to sum to n is the same as three of $2, 4, \cdots, n-4k-1$ to sum to $n-3k$, or, i.e., for three of $1, 2, \cdots, (n-4k-1)/2$ to sum to $(n-3k)/2$. Since $(n-4k-1)/2 \leqq \frac{2}{3}(n-3k)+2$, we find we are facing exactly the same problem as before with $(n-3k)/2$ replacing n. We may thus repeat our construction over and over again. To record the number of triples thus produced, call $N(n)$ the maximum number of triples again and conclude that

$$N(n) \geqq k + N\left(\frac{n-3k}{2}\right).$$

Since $k > (n/6)-2$, we may iterate this to give

$$N(n) \geqq \left(\frac{n}{6}-2\right)+\left(\frac{n}{6\cdot 4}-2\right)+\left(\frac{n}{6\cdot 4^2}-2\right)+\cdots+\left(\frac{n}{6\cdot 4^{j-1}}-2\right).$$

Finally, choosing $j=[\log_4 n/3]$ gives

$$N(n) \geqq \frac{2}{9}n - 2\log_4\frac{n}{3}\cdot$$

The next problem began in a supermarket! Slips of paper with positive integers imprinted are given to customers. When three of these integers add up to 100 the customer is a winner.

Question. How to sabotage the customers. Namely, what numbers can be removed so that a sum of 100 becomes impossible? Clearly the removal of 1, 2, 3, $\cdots$, 33 makes it impossible for three to sum to 100, but is this a minimal removal? Could not less than 33 other numbers be used? This leads to the following problem:

PROBLEM 3. *Produce a minimal set of positive integers such that k positive integers outside this set cannot add up to n (repetitions are allowed).*

We prove the following:

THEOREM. *The minimal set of positive integers such that k positive integers outside this set cannot add up to n if repetitions are allowed consists of $[n/k]$ integers.*

Proof. The construction is trivial; just take the integers 1, 2, 3, $\cdots$, $[n/k]$. To prove that we cannot do with fewer consider the k-tuples

$$\{1, 1, \cdots, 1, n-(k-1)\}, \{2, 2, \cdots, 2, n-(k-1)2\},$$

$$\cdots, \left\{\left[\frac{n}{k}\right], \cdots, \left[\frac{n}{k}\right], n-(k-1)\left[\frac{n}{k}\right]\right\}.$$

These are disjoint from one another and each has sum n. To block all k-tuples with sum n, then, one element must be chosen from each of these k-tuples. Hence at least $[n/k]$ elements must be so chosen and the proof is complete.

Let us now consider this same problem when repetitions are *not* allowed. Thus to block all triples summing to 100, we can take the set 1, 2, 3, $\cdots$, 32 (33 is not necessary this time).

In general, let us say that a set of integers is an (n, k) set if it intersects every k-tuple of distinct positive integers summing to n (e.g., $\{1, 2, 3, \cdots, 32\}$ is a (100, 3) set). We will prove the following (assuming $k>1$):

THEOREM. *The smallest (n, k) set has $[(n/k)-(k-1)/2]$ elements (e.g., the set $\{1, 2, \cdots, 32\}$ is smallest possible).*

Proof. Since $\{1, 2, \cdots, [(n/k)-(k-1)/2]\}$ is easily shown to be an (n, k) set, we need only show that no smaller set will do. The proof will be by double induction. To begin the induction, we observe that the result is trivial for $k=2$ and also for $n<k(k+1)/2$. We may assume the result for $k-1$ and all n, and also for k and all numbers below n.

Let S be an (n, k) set and suppose $n \geqq k(k+1)/2$. From S form the set T by subtracting 1 from each element, dropping the 0 if necessary.

Case I. $1 \notin S$. Let $(m_1, m_2, \cdots, m_{k-1}, 1)$ be any k-tuple of distinct positive integers ending in 1 and summing to n. Since S intersects it, S must intersect $(m_1, m_2, \cdots, m_{k-1})$ and so T must intersect $(m_1-1, m_2-1, \cdots, m_{k-1}-1)$. But $(m_1-1, m_2-1, \cdots, m_{k-1}-1)$ is an arbitrary $(k-1)$-tuple summing to $n-k$ and so T must be an $(n-k, k-1)$ set. By the inductive hypothesis, then,

$$|T| \geqq \left[\frac{n-k}{k-1} - \frac{k-2}{2}\right].$$

But $n \geqq k(k+1)/2$ and so $(n-k)/(k-1)-(k-2)/2 \geqq (n/k)-(k-1)/2$. Thus $|S| = |T| \geqq [(n/k)-(k-1)/2]$, as required.

Case II. $1 \in S$. T is, of course, an $(n-k, k)$ set since S is an (n, k) set. Hence by the inductive hypothesis,

$$|T| \geqq \left[\frac{n-k}{k} - \frac{k-1}{2}\right] = \left[\frac{n}{k} - \frac{k-1}{2}\right] - 1,$$

and the result follows since in this case $|S| = |T| + 1$.

As our final problem we take one which has no apparent connection with arithmetic. The statement is of a purely combinatorial nature, but our solution connects it with the previous arithmetical problems.

PROBLEM 4. *Given n objects, what is the largest number of triples of these objects such that any two of them overlap in at most one element?*

We prove that this maximum number is asymptotic to $(n^2/6)$. Construction: Number the objects $1, 2, \cdots, n$ and then choose all the triples whose numbers add up to 0 (mod n). These triples can have at most 1 element in common for if they had 2 then the third ones would be congruent mod n and so equal, and then the triples would be the same. To estimate the number of these triples, merely consider all solutions to $x+y+z=0$ (mod n) with x, y, z, all incongruent. We choose x arbitrarily and then y so as not to be x, $-\frac{1}{2}x$, or $-2x$. The value of z is then determined. This forces x, y, z to be pairwise incongruent. The total number of choices is thus at least $n(n-3)$ and the number of triples is, therefore, at least $n(n-3)/6$.

Proof. We now show that the maximum number of triples is bounded by $n(n-1)/6$. Namely, each triple contains three different couples [e.g., (a, b, c) contains (a, b), (b, c), and (c, a)] all of which must be different, for if two were the same, then the corresponding triples would have had this couple in common, contrary to hypothesis. But the total number of couples available is only $(n/2)$, and since each triple accounts for three of them, the number of the triples is bounded by $\frac{1}{3}(n/2) = n(n-1)/6$.

We mention in passing that attempts to generalize this last problem lead to interesting questions concerning symmetric functions. Typical among these questions is the following:

Can one find two symmetric functions $F(x, y, z, w)$ and $G(x, y, z, w)$ such that for every z, w there exists one and, up to symmetry, only one solution x, y to the system $F=0$, $G=0$?

THE NOTION OF "CONSEQUENCE" IN THE PREDICATE CALCULUS

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The purpose of this note is to investigate the technical notion of *consequence* (deducibility from assumptions) in the lower predicate calculus, to show that the definition presented in [1] is inadequate, and to present a workable formulation of this concept. The terminology and symbolism used here is that of [1]. A key to the notion of *consequence* is contained in the important *extended completeness theorem* which asserts that a nonempty set of wff is consistent iff it possesses a model. Now, this statement is equivalent to the following statement, where "K" denotes any nonempty set of wff.

(1) A is a consequence of K iff μA holds in $\mathcal{S}$ whenever $\mathcal{S}$ is a model of K under μ and μA is a swff of $\mathcal{S}$.

The right-hand side of (1) is to be interpreted in the following sense:

(2) μA holds in $\mathcal{S}$ whenever $\mathcal{S}$ is a structure and μ is a mapping such that:
   a. $\mathcal{S}$ is a model of K under μ.
   b. μA is a swff of $\mathcal{S}$.

Our point is that the notion of *consequence* must be defined so that it satisfies (1). Moreover, the notion of consequence must be such that *each* wff is a consequence of a set of the form $\{A, \sim A\}$. Now, the definition presented in [1], namely "A is a consequence of K iff there exists a nonempty finite subset of K, say $\{A_1, \cdots, A_n\}$, such that $\vdash A_1 \wedge \cdots \wedge A_n \rightarrow A$", fails on both counts. This is due to the technicality under which an expression of the form "$B \vee C$", where B and C are wff, is a wff iff no individual is free in one of B or C, and is bound in the other. For example, Fx is *not* a consequence of $\{\forall x Fx\}$ under this definition, yet (2) is true for $A = Fx$ and $K = \{\forall x Fx\}$. Moreover, under this definition, Fx is *not* a consequence of $\{\forall x Fx, \sim\forall x Fx\}$, for the same reason.

As a first suggestion toward resolving this difficulty, let a wff B be a consequence of K, a nonempty set of wff, iff there are wff $A_1, \cdots, A_n, A$ such that $\{A_1, \cdots, A_n\} \subset K$, $\vdash A_1 \wedge \cdots \wedge A_n \rightarrow A$, and $\vdash A \rightarrow B$. Consider the example given above; under this definition Fx is a consequence of $\{\forall x Fx\}$, since $\vdash \forall x Fx \rightarrow \forall y Fy$ and $\vdash \forall y Fy \rightarrow Fx$. However, fresh difficulties now appear. A basic property of *consequence* is that a wff $A \wedge B$ is a consequence of K if each conjunct of $A \wedge B$ is a consequence of K. To establish this, we observe that there are wff $A_1, \cdots, A_m, A'$ and $B_1, \cdots, B_n, B'$ such that

$$\{A_1, \cdots, A_m, B_1, \cdots, B_n\} \subset K,$$

$\vdash A_1 \wedge \cdots \wedge A_m \rightarrow A'$, $\vdash A' \rightarrow A$, $\vdash B_1 \wedge \cdots \wedge B_n \rightarrow B'$, and $\vdash B' \rightarrow B$. To demonstrate that $A \wedge B$ is a consequence of K, we would like to argue as follows:

$$\vdash (A_1 \wedge \cdots \wedge A_m) \wedge (B_1 \wedge \cdots \wedge B_n) \rightarrow A' \wedge B' \tag{3}$$

and $\vdash A' \wedge B' \rightarrow A \wedge B$, so $A \wedge B$ is a consequence of K.

The fallacy in this argument is that the expression that appears in (3) need not be a wff, for the usual reason; for example, there may be an individual which is free in $A_1 \wedge \cdots \wedge A_m$ and is bound in $B_1 \wedge \cdots \wedge B_n$.

It is now clear that we must come to grips with the technicality involved. One possibility is to drop this restriction on wff, so increasing the set of wff. However, there are certain advantages to maintaining this condition on wff that we are not willing to relinquish so easily; so we shall not pursue this line. In support of this view we mention that the number of cases involved in establishing a theorem is sometimes reduced by this convention (see [1], Theorem 8, P. 171; The Substitution Theorem, P. 176; Corollary 2, P. 177; Theorem 12, P. 178; Theorem 13, P. 203). The fundamental idea is that it is easier to establish that all wff (or all provable wff) possess a stated property if there are fewer wff around.

Our basic approach is that a wff B is a consequence of K, a nonempty set of wff, iff there are wff $A_1, \cdots, A_n$ equivalent to members of K, and a wff A equivalent to B, such that $\vdash A_1 \wedge \cdots \wedge A_n \rightarrow A$. However, in order to focus on the main problem, which revolves around free and bound individuals and the possibility that an expression is not a wff, we shall take a special case of this idea as our definition of *consequence*, namely that the wff $A_1, \cdots, A_n$ are obtained from members of K by substituting for bound individuals, and A is obtained from B by substituting for bound individuals. We now make this *substituting* procedure precise, by introducing the *-operation.

DEFINITION 1. *Let C be a wff. Let C_1 be obtained from C by replacing a component of C of the form "$\forall tE$" by $\forall s(S_t^s E)$, where s is not free in C and is chosen so that C_1 is a wff. Let C_2 be obtained from C_1 by replacing a component of C_1 of the form "$\forall uD$" by $\forall v(S_u^v D)$, where v is not free in C_1 and is chosen so that C_2 is a wff. Each wff obtained by carrying out a finite number of these substitutions is denoted by "C^*".*

We shall interpret this definition so that it embraces the case in which no substitutions are carried out; thus C^* can be C whenever C is a wff.

LEMMA 1. $C^* \equiv C$ whenever C is a wff.

Demonstration. First, we observe that $\forall tE \equiv \forall s(S_t^s E)$ provided that "$\forall tE \rightarrow \forall s S_t^s E$" is a wff; so, by the Substitution Theorem (see [1]), each wff obtained in the process of constructing C^* is equivalent to the wff that precedes it in the chain. Since there is no conflict between free and bound individuals throughout this process, the transitivity of $\equiv$ applies; we conclude that $C^* \equiv C$.

We now present our definition of *consequence.*

DEFINITION 2. *A wff B is a consequence of a nonempty set of wff K, iff there is a finite subset of K, say $\{A_1, \cdots, A_n\}$, and there exist wff $A_1^*, \cdots, A_n^*, B^*$, obtained by applying the *-operation, such that $\vdash A_1^* \wedge \cdots \wedge A_n^* \rightarrow B^*$.*

As usual, we shall denote "B is a consequence of K" by writing "$K \vdash B$". Here are some examples.

Example 1. $\{\forall x Fx\} \vdash Fx$. Take $(\forall x Fx)^* = \forall y Fy$ and take $(Fx)^* = Fx$. Clearly, $\vdash \forall y Fy \rightarrow Fx$.

Example 2. $\{Fx, \sim Fx\} \vdash \forall x Fx$. Take $(Fx)^* = Fx$, $(\sim Fx)^* = \sim Fx$, and $(\forall x Fx)^* = \forall y Fy$. Clearly, $\vdash Fx \wedge \sim Fx \rightarrow \forall y Fy$.

We shall need the following lemma, which is based on the fact that the *-operation is *local* in its application.

LEMMA 2. $A^* \wedge B^* = (A \wedge B)^*$ *whenever* $A \wedge B$ and $A^* \wedge B^*$ *are wff.*

Demonstration. We interpret this lemma as follows. Let C be a wff obtained from $A \wedge B$ by carrying out a chain of substitutions for bound individuals in the manner of Definition 1. Then there exist wff D and E that are obtained from A and B respectively by carrying out a chain of substitutions for bound individuals in the manner of Definition 1, such that $D \wedge E = C$. Of course, the substitutions that produce D are precisely the substitutions on $A \wedge B$ localized to A; the substitutions that produce E are precisely the substitutions on $A \wedge B$ localized to B. This establishes our lemma.

We shall now show that the conjunction of consequences of K is also a consequence of K.

THEOREM 1. $K \vdash A \wedge B$ *provided that* $K \vdash A$, $K \vdash B$, *and* $A \wedge B$ *is a wff.*

Demonstration. By assumption, there are subsets of K, say $\{A_1, \cdots, A_m\}$ and $\{B_1, \cdots, B_n\}$, and wff A^* and B^* such that

(4) $$\vdash A_1^* \wedge \cdots \wedge A_m^* \rightarrow A^*, \quad \text{and}$$

(5) $$\vdash B_1^* \wedge \cdots \wedge B_n^* \rightarrow B^*.$$

It may be that there is a conflict between free and bound individuals; for example, there may be an individual t which is bound in (4) and is free in (5). In that case we alter the *-operation involved in (4) by substituting s for t, where s is an individual which does not occur in (4) or (5). So, we may as well assume that the *-operation has been carried out so that no individual is bound in (4) and is free in (5), or vice-versa.

It now follows that

$$(A_1^* \wedge \cdots \wedge A_m^*) \wedge (B_1^* \wedge \cdots \wedge B_n^*) \rightarrow A^* \wedge B^*$$

is a wff. Clearly, this wff is provable. So, by Lemma 2, $\vdash (A_1^* \wedge \cdots \wedge A_m^*) \wedge (B_1^* \wedge \cdots \wedge B_n^*) \rightarrow (A \wedge B)^*$; thus, $K \vdash A \wedge B$.

Here is another useful property of our notion.

THEOREM 2. $K \vdash B$ *whenever* $K \vdash A$ *and* $\vdash A \rightarrow B$.

Demonstration. By assumption, there is a finite subset of K, say $\{A_1, \cdots, A_n\}$, such that $\vdash A_1^* \wedge \cdots \wedge A_n^* \rightarrow A^*$. Now, $\vdash A \rightarrow B$; so, by Lemma 1, $\vdash A^* \rightarrow B^*$ no matter how B^* is obtained. Choose B^* so that $A_1^* \wedge \cdots \wedge A_n^* \rightarrow B^*$ is a wff (this may also involve altering the *-operation for some of the A_i's). Then $\vdash A_1^* \wedge \cdots \wedge A_n^* \rightarrow B^*$; so $K \vdash B$.

Next, we shall consider the connection between the extended completeness theorem and (1). Recall that a nonempty set of wff K is said to be *contradictory* iff each wff is a consequence of K, and that a nonempty set of wff is said to be *consistent* iff it is not contradictory. We shall need the following lemma.

LEMMA 3. $K \cup \{\sim A\}$ *is consistent if* A *is not a consequence of* K.

Demonstration. Assume that $K \cup \{\sim A\}$ is contradictory. Then $K \cup \{\sim A\} \vdash A$; so, there is a finite subset of K, say $\{A_1, \cdots, A_n\}$, such that $\vdash A_1^* \wedge \cdots \wedge A_n^* \wedge (\sim A)^* \rightarrow A^*$. So $\vdash A_1^* \wedge \cdots \wedge A_n^* \rightarrow \sim(\sim A)^* \vee A^*$.

Of course, the substitutions involved in producing $(\sim A)^*$ from $\sim A$, are not necessarily the same as the substitutions involved in producing A^* from A. Nevertheless, by Lemma 1, $\sim(\sim A)^* \equiv A$ and $A^* \equiv A$; so $\sim(\sim A)^* \vee A^* \equiv A^*$. So, by the substitution theorem, $\vdash A_1^* \wedge \cdots \wedge A_n^* \rightarrow A^*$. Thus, $K \vdash A$. This contradiction establishes our lemma.

Using the properties of *consequence* developed here, it is easy to verify that the extended completeness theorem and (1) are equivalent in the sense that each is deducible from the other. (See [1].)

Since the extended completeness theorem is correct, it follows that (1) is correct. So, the syntactical notion of *consequence* given in Definition 2 has succeeded in characterizing the semantical notion contained in (2). We mention that the term "deducibility" is sometimes used for this syntactical concept. Finally, we point out that our notion of "consequence" has the properties listed by Montague and Henkin in [2].

References

1. A. H. Lightstone, The Axiomatic Method, Prentice-Hall, Englewood Cliffs, N. J., 1964.

2. Richard Montague and L. A. Henkin, On the definition of "formal deduction," J. Symb. Logic, 21 (1956) 129–136.

A DOODLING PROBLEM INVOLVING THE DENSITY OF SEGMENT-GENERATED SETS OF POINTS IN REGIONS OF A PLANE

LARRY HEACOCK, San Francisco

Everyone who has doodled has undoubtedly drawn a polygon and then drawn all the line segments joining the vertices, then drawn all line segments among the vertices and the intersections formed by the previously drawn segments, and so on, until the figure became a mess and patience was exhausted. This paper will treat this topic in a formal manner and demonstrate that, for certain polygons, the set of intersection points is dense in certain regions of the plane, provided that the drawing process continues infinitely. Theorem 8 is the key theorem of this discussion, leading us to the important results, Theorem 10 and Theorem 11.

Notation. AB will denote the distance from A to B; $\overline{AB}$ will denote the segment joining A and B; $\overleftrightarrow{AB}$ will denote the line through A and B. When I say that C lies between A and B, I imply that C lies between A and B on line $\overleftrightarrow{AB}$.

DEFINITION. *For any set, S, of points in the plane, and any positive integer n, define $\overline{S}_n$ and $\overline{S}$ as follows*: $\overline{S}_n = S$ *for* $n=1$; $\overline{S}_n = \{x$: *x is a point in the plane and at least one of the following is true*: (1) $x \in \overline{S}_{n-1}$; (2) *x is the intersection of the seg-*

ments $\overline{AB}$ and $\overline{CD}$ where $A, B, C, D \in \bar{S}_{n-1}\}$. We will say that $\bar{S}=\bigcup_{n=1}^{\infty} \bar{S}_n$ is the set of points segment-generated by S.

THEOREM 1. *Let A, B, P, Q, E, F, G be distinct points in the plane such that:*
(1) *P is between E and G, and Q is between F and G.*
(2) *A and B are between P and Q.*
(3) *A is between P and B.*
(4) $0° < \angle EGF < 180°$.

Then there is a $k>0$ such that given any points, C and D, on $\overline{AB}$, such that $C \neq D$ and C is between A and D, there is a point I between C and D such that $I \in \bar{S}$, where $S=\{P, Q, E, F, G, C, D\}$, and such that $CI > k\, CD$ and $DI > kCD$.

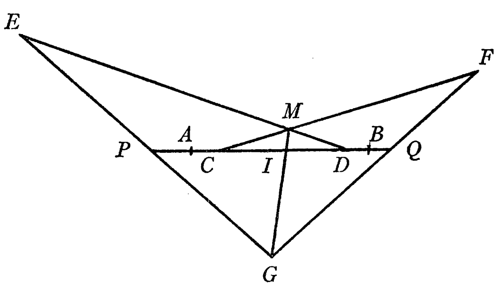

FIG. 1

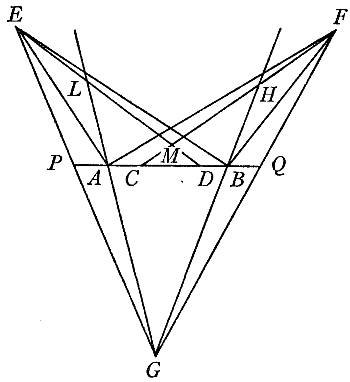

FIG. 2

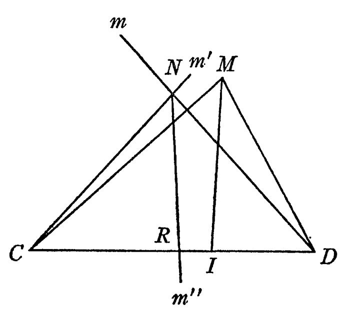

FIG. 3

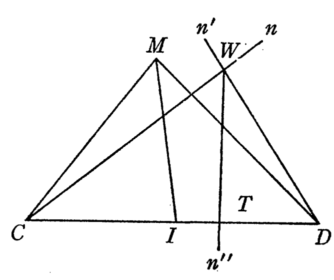

FIG. 4

Proof. Let A, B, P, Q, E, F, G be points meeting the requirements of our theorem (Figure 1). Let C and D be points on $\overline{AB}$ which meet the requirements set above. Construct $\overline{CF}$ and $\overline{ED}$, intersecting at M. Clearly $M \in \bar{S}$. Draw $\overline{GM}$, intersecting $\overline{CD}$ at I, between C and D. Clearly $I \in \bar{S}$. In Figure 2, draw $\overleftrightarrow{GA}$, intersecting $\overline{DE}$ at L; $\overleftrightarrow{GB}$ intersecting $\overline{CF}$ at H; draw $\overline{BE}$, $\overline{CF}$, $\overline{AE}$, and $\overline{BF}$. In Figure 3, draw line m through D, parallel to $\overline{BE}$. Draw m' through C, parallel to $\overline{BF}$. Let m intersect m' at N. Draw m'' through N, parallel to $\overline{BG}$. Let m'' intersect $\overline{CD}$ at R. Clearly $RC < CI$. Draw n through C, parallel to $\overline{AF}$ (Figure 4). Draw n' through D, parallel to $\overline{AE}$. Let n intersect n' at W. Draw n'' through

W, parallel to $\overline{AG}$. Let n'' intersect $\overline{CD}$ at T. We see that $TD<DI$. Elementary operations show that $CR = CD \sin(\angle ABE)\sin(\angle HBF)/\sin(\angle EBF)\sin(\angle ABH) = pCD$ and $TD = CD \sin(\angle BAF)\sin(\angle LAE)/\sin(\angle FAE)\sin(\angle BAL) = qCD$.

Thus $CI>RC=pCD$ and $DI>TD=qCD$. For fixed A, B, E, F, G, P, Q we see that p and q are fixed. Thus, if we let $k=\min(p, q)$ we see that $CI>kCD$, $DI>kCD$ for any C, D selected by our requirements. Thus k and I satisfy the theorem.

THEOREM 2. *Let A, B, P, Q, E, F, G be points satisfying the conditions of the previous theorem. Let $S=\{A, B, P, Q, E, F, G\}$. Then S is dense in $\overline{AB}$.*

FIG. 5

Proof. Let S, A, B, P, Q, E, F, G satisfy the conditions of Theorem 1 and suppose that Theorem 2 is not true. Thus there are points X and Y on $\overline{AB}$ (Figure 5) such that $X\neq Y$ and X is between A and Y, and such that if Z is between X and Y, then $Z\notin\overline{S}$. Let $t=\sup(\{AX' : X'\in\overline{S}$ and X' is between A and $X\})$ and let $s=\sup(\{BY' : Y'\in\overline{S}$ and Y' is between B and $Y\})$. Let X'', Y'' be those points between A and B such that $AX''=t$, $BY''=s$. Clearly $X''Y'' \geqq XY$. Now consider k as defined in Theorem 1 and choose d so that $k(X''Y'')>d$. There are $C, D\in\overline{S}$ between A and B such that $C\neq D$, C is between A and X'', D is between B and Y'', and $0\leqq CX''<d$. Construct I as in Theorem 1. Then $I\in\overline{S}$ and, by our supposition, I lies between C and X'' or else I lies between D and Y''. Because of the symmetry of the problem, we need only consider the case where I lies between C and X''. Now, $kCD\geqq k(X''Y'')$. If I lies between C and X'', then $CI<CX''<d<kCD$. However, by the previous theorem, $CI>kCD$. A contradiction! Therefore, our supposition is incorrect; i.e., $\overline{S}$ is dense in $\overline{AB}$.

THEOREM 3. *If A, B, P, Q, E, F, G are points satisfying the conditions of the above theorems and if $S=\{A, B, P, Q, E, F, G\}$, then $\overline{S}$ is dense in $\overline{AP}$ and $\overline{BQ}$.*

Proof. Because of the symmetry of the problem we need only show the theorem true for $\overline{AP}$. Suppose $\overline{S}$ is not dense in $\overline{AP}$. Then $A\neq P$. Let $t=\inf(\{PX' : X'\in\overline{S}\})$. Clearly $t>0$. Let X be that point between P and B such that $PX=t$ (Figure 6). Draw $\overleftrightarrow{EX}$ intersecting $\overline{PF}$ at M; $\overleftrightarrow{GX}$ intersecting $\overline{PF}$ at

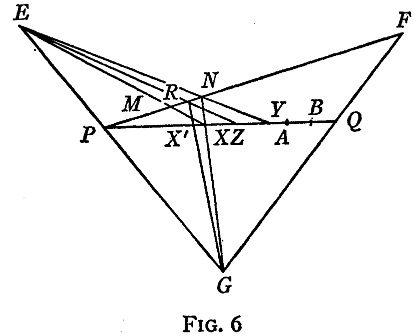

FIG. 6

N. Draw $\overleftrightarrow{EN}$ intersecting $\overleftrightarrow{PQ}$ at Y. Clearly X is between P and Y. Also, there is a point $Z \in \overline{S}$ such that Z is between X and Y. Draw $\overline{EZ}$ intersecting $\overline{PF}$ at R. Clearly $R \in \overline{S}$ and R is between M and N and $R \neq N$. Draw $\overline{RG}$ intersecting $\overline{PQ}$ at X'. Clearly X' is between P and X, and $X' \in \overline{S}$, and $X' \neq X$. That is, $PX' < PX$, which is a contradiction. Therefore, our supposition is incorrect, and $\overline{S}$ is dense in $\overline{PA}$.

THEOREM 4. *If A, B, P, Q, E, F, G are points satisfying the three preceding theorems, then $\overline{S}$ is dense in $\overline{PQ}$, where $S = \{A, B, P, Q, E, F, G\}$.*

Proof. The two preceding theorems prove this theorem.

THEOREM 5. *Let G, X, E, F, Y be five distinct points on a circle, lettered clockwise in order (Figure 7). Draw $\overline{GE}$ and $\overline{GF}$ and $\overline{XY}$, letting $\overline{XY}$ intersect $\overline{GE}$ at P and $\overline{GF}$ at Q. Then $\overline{S}$ is dense in $\overline{PQ}$ where $S = \{G, X, E, F, Y\}$, and $P \neq Q$.*

Proof. Obviously P, Q are distinct from G, X, Y, E, F and $P \neq Q$. Form $\overline{PF}$ and $\overline{EQ}$, intersecting at M; $\overline{GM}$ intersecting $\overline{PQ}$ at A; $\overline{AF}$ intersecting $\overline{EQ}$ at N; $\overline{GN}$ intersecting $\overline{PQ}$ at B. Then A, B, P, Q, E, F, G satisfy the above theorem. Therefore $\overline{T}$ is dense in $\overline{PQ}$, where $T = \{A, B, P, Q, E, F, G\}$. However, $\overline{T} \subseteq \overline{S}$. Hence $\overline{S}$ is dense in $\overline{PQ}$.

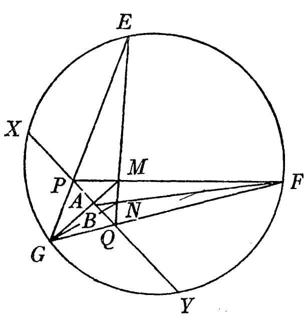

FIG. 7

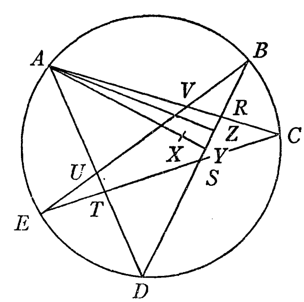

FIG. 8

THEOREM 6. *Let A, B, C, D, E be five distinct points on a circle, lettered clockwise in order. Draw $\overline{AC}$, $\overline{BD}$ intersecting $\overline{AC}$ at R; $\overline{CE}$ intersecting $\overline{BD}$ at S; $\overline{DA}$ intersecting $\overline{CE}$ at T; $\overline{EB}$ intersecting $\overline{AD}$ at U and $\overline{AC}$ at V. Then, given any point X in the region $RSTUV$, and given any $d > 0$, there is a $Z \in \overline{M}$, where $M = \{A, B, C, D, E\}$, such that the distance of X from $\overleftrightarrow{AZ}$ is less than d.*

Proof. Form the figure as required (Figure 8) and let X be any point in the region $RSTUV$. Clearly $\overleftrightarrow{AX}$ intersects $\overline{RS}$ or $\overline{ST}$ at a point; call it Y. Because of the symmetry of the problem, we need only consider the case where $\overleftrightarrow{AX}$ intersects $\overline{RS}$. By Theorem 5, we know that $\overline{M}$ is dense in $\overline{RS}$. Therefore, let Z be on $\overline{RS}$, $Z \in \overline{M}$, such that $YZ < d$ and such that X lies in the exterior of $\angle CAZ$. It is apparent that the distance, d', of Y to $\overleftrightarrow{AZ}$ is greater than or equal to the distance, d'', of X to $\overleftrightarrow{AZ}$. And $YZ \geqq d'$. Thus $d'' \leqq d' \leqq YZ < d$. That is, the distance of X from $\overleftrightarrow{AZ}$ is less than d.

THEOREM 7. *Let A, B, C, D, E, R, S, T, U, V and M be defined as in Theorem 6, and let X be any point in the region $RSTUV$. Then $\overline{M}$ is dense in region $RSTUV$.*

Proof. Form the required figure (Figure 9). Let $d>0$ be chosen and let X be an arbitrarily chosen point in the region $RSTUV$. Then, by the previous theorem, for any $k>0$ there are $Y, Z\in\overline{M}$, such that the distance of X from $\overleftrightarrow{CY}$ and $\overleftrightarrow{DZ}$ is less than k and X lies in the exterior of $\angle ACY$ and $\angle ADZ$ (Figure 10). Note that Z lies on $\overline{UV}$ or $\overline{VR}$, and Y lies on $\overline{TU}$ or $\overline{UV}$. Let W be the intersection of $\overleftrightarrow{CY}$ and $\overleftrightarrow{DZ}$. Clearly $W\in\overline{M}$. Let line m be drawn through X and perpendicular to $\overleftrightarrow{CY}$, intersecting $\overleftrightarrow{CY}$ at H. Draw line m' through X and perpendicular to $\overleftrightarrow{DZ}$, intersecting $\overleftrightarrow{DZ}$ at J. Now $\angle CAD\leqq\angle CWD\leqq\angle CSD$. Let's find WX. First we find HJ. We have $(HJ)^2=(HX)^2+(XJ)^2-2(HX)(XJ)\cos(180^\circ-\angle HWJ)=(HX)^2+(XJ)^2+2(HX)(XJ)\cos(\angle HWJ)$; $(WX)^2=(WJ)^2+(XJ)^2$ and $WJ=(HJ)\sin(\angle WHJ)/\sin(\angle HWJ)$.

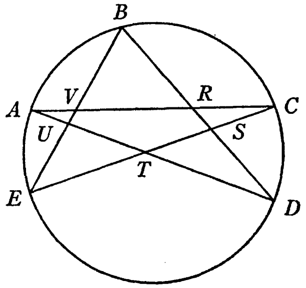

FIG. 9

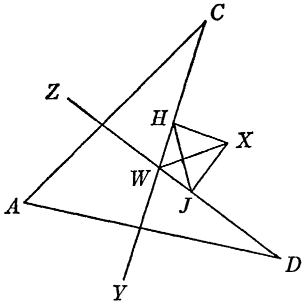

FIG. 10

Thus $(WX)^2=\sin^2(\angle WHJ)\sin^{-2}(\angle HWJ)(HJ)^2+(XJ)^2$. Also, $\angle WHJ=90^\circ-\angle JHX$. Hence $\sin(\angle WHJ)=\sin(90^\circ-\angle JHX)=\cos(\angle JHX)$. Therefore $(WX)^2=\cos^2(\angle JHX)\sin^{-2}(\angle HWJ)(HJ)^2+(XJ)^2=(1-\sin^2(\angle JHX))\sin^{-2}(\angle HWJ)(HJ)^2+(XJ)^2$. Now $\sin(\angle JHX)=\sin(\angle HXJ)(XJ)/(HJ)=\sin(180^\circ-\angle HWJ)(XJ)/(HJ)=\sin(\angle HWJ)(XJ)/(HJ)$. Therefore $(WX)^2=(1-\sin^2(\angle HWJ)(XJ)^2(HJ)^{-2})\sin^{-2}(\angle HWJ)(HJ)^2+(XJ)^2=(\sin^{-2}(\angle HWJ)-(XJ)^2(HJ)^{-2})(HJ)^2+(XJ)^2=(HJ)^2\sin^{-2}(\angle HWJ)$.

Let $p=\min(\sin(\angle CSD), \sin(\angle CAD))$. We see that $\sin(\angle HWJ)\geqq p$. Thus $(WX)^2\leqq(HJ)^2/p^2\leqq((HX)+(XJ))^2/p^2<4k^2/p^2$. Consequently, $WX<2k/p$. Let $k<pd/2$. It then follows that $W\in\overline{M}$ and $WX<d$.

It should be obvious that $RSTUV$ is the largest region in the plane in which $\overline{M}$ is dense.

THEOREM 8. *If A, B, C, D, E are any five distinct points on a nondegenerate conic with eccentricity $\leqq 1$, then, letting R, S, T, U, V be found as above, it follows that $\overline{M}$ is dense in $RSTUV$, where $M=\{A, B, C, D, E\}$.*

Proof. If A, B, C, D, E are any five distinct points on a nondegenerate conic with eccentricity $\leqq 1$, then there is a cone containing this section and a circle is the central projection, f, of this section onto a plane perpendicular to the axis but not passing through the focus of the cone. The focus of the cone is taken as the center of the projection. Since we have chosen a nondegenerate case, f^{-1} exists, and f and f^{-1} preserve distinctness of line and point and preserve density. Let $f(X)=X'$ for each X. Then, by Theorem 7, $\overline{M}'$ is dense in $R'S'T'U'V'$ in the plane of the circle. Applying f^{-1} we find that $\overline{M}$ is dense in $RSTUV$.

The converse of Theorem 8 is not true. For example, consider a degenerate conic consisting of two parallel lines with A, B, C on one line and D, E on the other. Let $S=\{A, B, C, D, E\}$. However, we do have the following interesting but trivial result:

THEOREM 9. *A set S must contain at least five points if $\overline{S}$ is to be dense in a region of the plane.*

Proof. If S contains four or fewer points, then $\overline{S}$ is finite.

Since five points in a plane determine a conic section we may use Theorem 8 to prove:

THEOREM 10. *If A, B, C, D, E are any five distinct points in a plane and determine a nondegenerate conic section such that they all lie on one branch of the conic, then letting R, S, T, U, V be found as above, it follows that $\overline{M}$ is dense in $RSTUV$, where $M=\{A, B, C, D, E\}$.*

Proof. Because of Theorem 8, we need only consider the case where the points determine a hyperbola. Given such conditions we know that there is a cone containing this conic section and that the section may be orthogonally projected onto a parabolic section of this cone. This projection, f, preserves distinctness of line, point, and density, as does its inverse, f^{-1}. With $f(X)=X'$ we see by Theorem 8 that $\overline{M}'$ is dense in $R'S'T'U'V'$ in the plane of the parabola. Thus, applying f^{-1}, we see that $\overline{M}$ is dense in $RSTUV$.

These results may be extended still further.

DEFINITION. *For any set S of points in the plane, and any positive integer n, define S_n^* and S^* as follows: $S_n^*=S$ for $n=1$; $S_n^*=\{x: x$ is a point in the plane and at least one of the following is true: (1) $x\in S_{n-1}^*$; (2) x is the intersection of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ where $A, B, C, D\in S_{n-1}^*\}$. We will say that $S^*=\bigcup_{n=1}^{\infty} S_n^*$ is the set of intersect-generated points generated by S. Clearly $\overline{S}\subseteq S^*$.*

THEOREM 11. *Let A, B, C, D, E be five distinct points in a plane which determine a nondegenerate conic and lie on one branch of the conic. Then S^* is dense in the plane, where $S=\{A, B, C, D, E\}$.*

Proof. Form R, S, T, U, V as shown in Figure 11. By Theorem 10 we know $\overline{S}$ is dense in $RSTUV$. Let X be any point in the plane and let $d>0$ be chosen. Let $Y, Z\in\overline{S}$ and such that X, Y, Z are not collinear. Let Y', Z' be points in region $RSTUV$ such that Y' is between Y and X, Z' is between Z and X. If $k>0$ there are $Y'', Z''\in\overline{S}$ such that $Y'Y''<k$, $Z'Z''<k$, and Y'', Z'' are in the interior of $\angle YXZ$. Let $\overleftrightarrow{ZZ''}$ intersect $\overleftrightarrow{YY''}$ at X'. Draw m through X' and perpendicular to $\overleftrightarrow{XY}$, intersecting $\overleftrightarrow{XY}$ at H. Draw m' through X' and perpendicular to $\overleftrightarrow{XZ}$, intersecting $\overleftrightarrow{XZ}$ at J. Some manipulations show that $XX'=HJ/|\sin(\angle HXJ)| \leqq (HX'+JX')/|\sin(\angle HXJ)| <2k/|\sin(\angle HXJ)|$. Since $\angle HXJ$ is constant we may let $k<d|\sin(\angle HXJ)|/2$. Consequently $XX'<d$ and, of course, $X'\in S^*$.

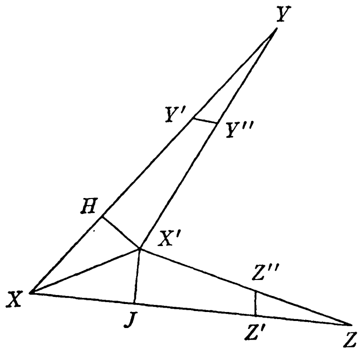

FIG. 11

Returning to polygons we see that if any five of the vertices of a polygon determine a nondegenerate conic, and lie on one branch of that conic, then the set of segment-generated points generated by the vertices is dense in a specifiable region of the plane. Also, the set of intersect-generated points is dense in the plane. Thus our doodling conjecture is decided.

CONSTRUCTIONS FOR THE SOLUTION OF THE m QUEENS PROBLEM

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The problem of the m queens, originally introduced by Gauss (with $m=8$), may be stated as follows: is it possible to place m queens on an $m\times m$ chessboard so that no one queen can be taken by any other? The problem is an interesting one because it reduces to that of finding a maximum internally stable set, S, of a symmetric graph, $G=(X, \Gamma)$, the vertices of which correspond to the m^2 square elements of an $m\times m$ matrix, where x' is an element of Γx only if x and x' are on the same row or column or diagonal, and where $\Gamma S\cap S$ is the null set. (See [1].) Obviously, $|S|$ cannot be greater than m.

By treating the chessboard as an $m\times m$ matrix of square elements, we can identify any square by an ordered pair, (i, j), where i and j are the row and column numbers of the square, respectively. We define a *major diagonal* of the matrix to be a set of elements (i, j) such that $m-j+i=$ CONSTANT where the CONSTANT is the number of the diagonal. The major diagonal numbered m will be called the *principal diagonal*. Clearly, all points on the principal diagonal have the property $i=j$.

We further define a *minor diagonal* of the matrix to be a set of elements (i, j) such that $i+j-1=$ CONSTANT where the CONSTANT is the number of the diagonal.

The m queens problem can now be stated as follows: place m queens on an $m \times m$ matrix of square elements so that, for the elements occupied,

a) the row numbers are unique,
b) the column numbers are unique,
c) the major diagonal numbers are unique, and
d) the minor diagonal numbers are unique.

The constructions which follow are sufficient to solve the m queens problem; the theorems delineate which of the constructions are appropriate for a given m. It will be shown that the solutions apply for all $m \geqq 4$.

Construction A. Form an $m \times m$ matrix of square elements with $m = 2n$, where $n = 2, 3, 4, 5, \cdots$.

i) Place queens on the elements (i_k, j_k), where $i_k = k$ and $j_k = 2k$, $k = 1, 2, 3, \cdots, n$.

ii) Place queens on the elements (i_l, j_l), where $i_l = 2n+1-l$ and $j_l = 2n+1-2l$, $l = 1, 2, 3, \cdots, n$.

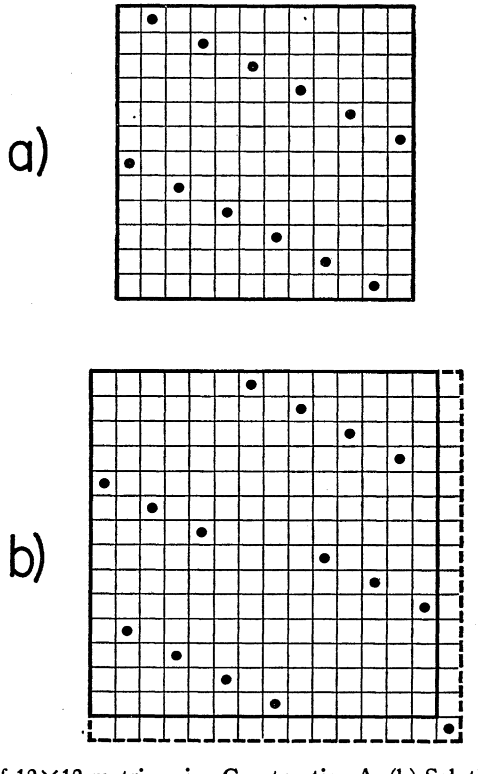

FIG. 1. (a) Solution of 12×12 matrix using Construction A. (b) Solution of 14×14 matrix using Construction B and extension to 15×15 matrix using Construction C.

Construction B. Form an $m \times m$ matrix of square elements with $m = 2n$, where $n = 2, 3, 4, 5, \cdots$.

i) Place queens on the elements (i_k, j_k), where $i_k = k$ and

$$j_k = 1 + \{[2(k-1) + n - 1] \text{ modulo } m\}, \qquad k = 1, 2, 3, \cdots, n.$$

ii) Place queens on the elements (i_l, j_l), where $i_l = 2n+1-l$ and $j_l = 2n - \{[2(l-1)+n-1] \text{ modulo } m\}$, $l = 1, 2, 3, \cdots, n$.

Construction C. To an $m \times m$ matrix of square elements add an $(m+1)$th row and an $(m+1)$th column. Place a queen on the element $(m+1, m+1)$.

Figure 1 shows typical examples of Constructions A, B, and C.

THEOREM 1. *A solution of the m queens problem is obtained when Construction* A *is applied to an* $m \times m$ *matrix,* $m = 2n$, *where n is an integer greater than zero such that* $n \neq 3\lambda + 1$, $\lambda = 0, 1, 2, \cdots$.

Proof. Part i) of Construction A places queens on the elements $(k, 2k)$ while part ii) places queens on the elements $(2n+1-l, 2n+1-2l)$, $1 \leqq (k, l) \leqq n$. Clearly, part i) places one queen on an element of each of the first n rows and also on each even-numbered column. Part ii) places one queen on an element of each of the second n rows and also on each odd-numbered column. Therefore, each row and column has one and only one queen.

The major diagonals which are used by part i) are numbered $2n - 2k + k = 2n - k$, $1 \leqq k \leqq n$. Clearly, these are unique. The major diagonals used by part ii) are numbered $2n - (2n+1-2l) + 2n + 1 - l = 2n + l$, $1 \leqq l \leqq n$. Clearly, these are also unique.

Assume a queen from part i) occupies the same major diagonal as a queen from part ii). Then $2n - k = 2n + l$ and $-k = l$ which is impossible, so we are forced to abandon the hypothesis that two queens occupy the same major diagonal.

The minor diagonals which are used by part i) are numbered $k + 2k - 1 = 3k - 1$, $1 \leqq k \leqq n$. Clearly, these are unique. The minor diagonals used by part ii) are numbered $2n+1-l+2n+1-2l-1 = 4n - 3l + 1$, $1 \leqq l \leqq n$. Clearly, these are also unique.

Assume a queen from part i) occupies the same minor diagonal as a queen from part ii) so that $3k - 1 = 4n - 3l + 1$ and $4n = 3(k+l) - 2$. Since n is an integer, $k + l$ must be even, and we can write

$$2n = 3\left(\frac{k+l}{2}\right) - 1.$$

Now we see that $(k+l)/2$ must be odd, say, $(k+l)/2 = 2\lambda + 1$, $\lambda = 0, 1, 2, \cdots$ and we have $2n = 3(2\lambda+1) - 1 = 6\lambda + 2$ so that $n = 3\lambda + 1$, $\lambda = 0, 1, 2, \cdots$, which are the values excluded by the theorem. Hence we are forced to abandon the hypothesis that two queens occupy the same minor diagonal, and the theorem is established.

THEOREM 2. *A solution of the m queens problem is obtained when Construction B is applied to an* $m \times m$ *matrix,* $m = 2n$, *where n is an integer greater than unity such that* $n \neq 3\lambda$, $\lambda = 1, 2, 3, \cdots$.

Proof. Part i) of Construction B places queens on the elements $(1, n)$, $(2, n+2)$, $(3, n+4)$, $\cdots$, (r, s) where

$$r = \begin{Bmatrix} \dfrac{n+2}{2}, & n \text{ even} \\ \dfrac{n+1}{2}, & n \text{ odd} \end{Bmatrix} \text{and } s = \begin{Bmatrix} 2n, & n \text{ even} \\ 2n-1, & n \text{ odd} \end{Bmatrix},$$

and on the elements (r', s'), $(r'+1, s'+2)$, $\cdots$, $(n, n-2)$ where

$$r' = r + 1 \quad \text{and} \quad s' = \begin{Bmatrix} 2, & n \text{ even} \\ 1, & n \text{ odd} \end{Bmatrix}.$$

Part ii) of Construction B places queens on the elements $(2n, n+1)$, $(2n-1, n-1)$, $(2n-2, n-3)$, $\cdots$, (p, q) where

$$p = \begin{Bmatrix} \dfrac{3n}{2}, & n \text{ even} \\ \dfrac{3n+1}{2}, & n \text{ odd} \end{Bmatrix} \text{and } q = \begin{Bmatrix} 1, & n \text{ even} \\ 2, & n \text{ odd} \end{Bmatrix},$$

and on the elements (p', q'), $(p'-1, q'-2)$, $(p'-2, q'-4)$, $\cdots$, $(n+1, n+3)$ where

$$p' = p - 1 \quad \text{and} \quad q' = \begin{Bmatrix} 2n-1, & n \text{ even} \\ 2n, & n \text{ odd} \end{Bmatrix}.$$

Clearly, part i) places one queen on an element of each of the first n rows, and also on each even-numbered column (if n is even) or each odd-numbered column (if n is odd). Part ii) places one queen on an element of each of the second n rows and also on each odd-numbered column (if n is even) or each even-numbered column (if n is odd). Therefore, each row and column has one and only one queen.

The major diagonals which are used by part i) are numbered $2n-[2(k-1)+n]+k=n-k+2$ for $1 \leqq k \leqq r$, and $2n-[2(k'-1)-n]+k'=3n-k'+2$ for $r' \leqq k' \leqq n$. Clearly, since the largest of the first set is $n-1+2=n+1$ and the smallest of the second set is $3n-n+2=2(n+1)$, these are unique. The major diagonals used by part ii) are numbered

$$2n - \{2n + 1 - [2(l-1) + n]\} + 2n + 1 - l = 3n + l - 2 \qquad 1 \leqq l \leqq 2n + 1 - p,$$

and

$$2n - \{2n + 1 - [2(l'-1) - n]\} + 2n + 1 - l' = n + l' - 2 \qquad 2n + 1 - p' \leqq l' \leqq n.$$

Clearly, since the smallest of the first set is $3n+1-2=3n-1$ and the largest of the second set is $n+n-2=2(n-1)$, these are unique.

Assume a queen from part i) occupies the same major diagonal as a queen from part ii). We would then have

(1) $n-k+2=3n+l-2$, (3) $3n-k'+2=3n+l-2$, or

(2) $n-k+2=n+l'-2$, (4) $3n-k'+2=n+l'-2$.

Equation (1) implies $k+l=4-2n$, but since the smallest $k+l$ can be is 2, this can never happen, since n is greater than unity. Equation (2) implies $k+l'=4$, and the smallest $k+l'$ will ever be is

$$\left\{\begin{matrix} \dfrac{n+6}{2}, & n \text{ even} \\ \dfrac{n+5}{2}, & n \text{ odd} \end{matrix}\right\}.$$

Equation (3) implies $k'+l=4$, and the smallest $k'+l$ will ever be is

$$\left\{\begin{matrix} \dfrac{n+6}{2}, & n \text{ even} \\ \dfrac{n+5}{2}, & n \text{ odd} \end{matrix}\right\}.$$

Equations (2) and (3) can be satisfied, therefore, only if $n=2$ or $n=3$. The value $n=3$ is excluded in the statement of the theorem. For $n=2$ we have $r=2$ and $2n+1-p=n$ so that $r'=r+1>n$ and $2n+1-p'=2n+2-p>n$, and neither k' nor l' can exist. Equation (4) implies $k'+l'=2n+4$, but the largest $k'+l'$ will ever be is $2n$, so we are forced to abandon the hypothesis that two queens occupy the same major diagonal.

The minor diagonals which are used by part i) are numbered $k+2(k-1)+n-1=n+3k-3$ for $1\leqq k\leqq r$, and $k'+2(k'-1)-n-1=-n+3k'-3$ for $r'\leqq k'\leqq n$. The minor diagonals used by part ii) are numbered $2n+1-l+2n+1-2(l-1)-n-1=3n-3l+3$ for $1\leqq l\leqq 2n+1-p$, and $2n+1-l'+2n+1-2(l'-1)+n-1=5n-3l'+3$ for $2n+1-p'\leqq l'\leqq n$.

Assume two queens occupy the same minor diagonal. We would then have

(5) $n+3k-3=-n+3k'-3$, (8) $-n+3k'-3=3n-3l+3$,

(6) $n+3k-3=3n-3l+3$, (9) $-n+3k'-3=5n-3l'+3$, or

(7) $n+3k-3=5n-3l'+3$, (10) $3n-3l+3=5n-3l'+3$.

Equation (5) implies $2n=3(k'-k)$, so $k'-k$ must be even, say $k'-k=2\lambda$, $\lambda=1, 2, 3, \cdots$. Then $2n=3(2\lambda)$, $n=3\lambda$, which are the values excluded by the theorem. Similarly, equation (10) results in $2n=3(l'-l)$, $l'-l=2\lambda$, $n=3\lambda$.

Equation (6) implies $2n=3(k+l-2)$, so $k+l-2$ must be even, say $k+l-2=2\lambda$, $\lambda=1, 2, 3, \cdots$. Then $2n=3(2\lambda)$, $n=3\lambda$, which are the values excluded by the theorem.

Equation (7) implies $4n=3(k+l'-2)$, so $k+l'-2$ must be doubly even, say $k+l'-2=4\lambda$, $\lambda=1, 2, 3, \cdots$. Then $4n=3(4\lambda)$, $n=3\lambda$, which are the values excluded by the theorem. Similarly, equation (8) results in $4n=3(k'+l-2)$, $k'+l-2=4\lambda$, $n=3\lambda$.

Finally, equation (9) implies $6n=3(k'+l'-2)$, $2n=k'+l'-2$. But the largest $k'+l'-2$ will ever be is $2n-2$, so we are forced to abandon the hypothesis that two queens occupy the same minor diagonal, and the theorem is established.

Before examining the validity of Construction C, it will be necessary to prove the following two lemmas:

LEMMA 1. *Construction* A *places no queens on the principal diagonal.*

Proof. The principal diagonal was defined as the major diagonal for which $i=j$. Suppose a queen from Construction A occupies the principal diagonal. Then either

(1) $k = 2k$ for some $1 \leqq k \leqq n$, or

(2) $2n + 1 - l = 2n + 1 - 2l$ for some $1 \leqq l \leqq n$.

Equation (1) implies $k=0$, which contradicts the bound $k\geqq 1$. Likewise, equation (2) implies $l=0$, contradicting the bound $l\geqq 1$. We therefore abandon the hypothesis that a queen exists on the principal diagonal, and the lemma is established.

LEMMA 2. *Construction* B *places no queens on the principal diagonal.*

Proof. The reasoning here is similar to that in Lemma 1. Suppose a queen from part (i) of Construction B occupies the principal diagonal. Then either

(1) $2(k - 1) + n = k$ for some $1 \leqq k \leqq r$, or

(2) $2(k' - 1) - n = k'$ for some $r' \leqq k' \leqq n$.

Equation (1) implies $k=2-n$. But $n>1$, implying $k\leqq 0$, which contradicts the bound $k\geqq 1$. Equation (2) implies $k'=n+2$, contradicting the bound $k'\leqq n$. Thus no queen from part (i) can occupy the principal diagonal. Suppose a queen from part (ii) occupies the principal diagonal. Then either

(3) $2n+1-[2(l-1)+n]=2n+1-l$ for some $1 \leqq l \leqq 2n+1-p$, or

(4) $2n+1-[2(l'-1)-n]=2n+1-l'$ for some $2n+1-p' \leqq l' \leqq n$.

Equation (3) implies $l=2-n$. But $n>1$, implying $l\leqq 0$, which contradicts the bound $l\geqq 1$. Equation (4) implies $l'=n+2$, contradicting the bound $l'\leqq n$. Thus no queen from part (ii) can occupy the principal diagonal, and the lemma is established.

THEOREM 3. *A solution of the m queens problem for an* $(m+1)\times(m+1)$ *matrix is obtained when Construction* C *is applied to an* $m\times m$ *matrix which has previously been solved by Construction* A *or Construction* B.

Proof. By creating a new $(m+1)$th row and $(m+1)$th column and placing a queen at $(m+1, m+1)$, Construction C obviously preserves unique row and column numbers. In addition, it creates a new minor diagonal containing only the element $(m+1, m+1)$, thus preserving unique minor diagonal numbers.

The principal diagonal of the $(m+1)\times(m+1)$ matrix is, however, an extension of the principal diagonal of the $m\times m$ matrix. But from Lemmas 1 and 2 we know that this diagonal must have been empty. Hence, unique major diagonal numbers are preserved and the theorem is established.

Each of the three constructions contained an exclusion for certain values of m. It remains only to prove

THEOREM 4. *The m queens problem is solved for all $m \geqq 4$ by either Construction* A, B, *or* C.

Proof. Construction A applies to even m except for $m=2(3\lambda+1), \lambda=0, 1, 2, \cdots$, or

$$m = 6\lambda_A - 4 \qquad \lambda_A = 1, 2, 3, \cdots . \tag{1}$$

Construction B applies to even m except for $m=2(3\lambda_B)$

$$m = 6\lambda_B \qquad \lambda_B = 1, 2, 3, \cdots , \tag{2a}$$

and

$$m = 2. \tag{2b}$$

The special case, equation (2b), results from the exclusion of $n=1$ in the statement of Theorem 2.

Finally, Construction C applies to all odd m for which $m-1$ can be solved by either A or B.

Let M' be the set of integers $m'>1$ having the property that neither Construction A, B, or C solves the $m'\times m'$ matrix. Any even member, m_e', of M' must be simultaneously excluded from both A and B, implying either

$$6\lambda_A - 4 = m_e' = 2, \quad \text{or} \tag{3}$$

$$6\lambda_A - 4 = m_e' = 6\lambda_B \tag{4a}$$

$$\lambda_A - 2/3 = \lambda_B \tag{4b}$$

for some pair of integers λ_A and λ_B. Equation (4) can never be satisfied by integer λ's. Equation (3) is satisfied only at $\lambda_A=1$, so that $m_e'=2$ is the only even member of M'.

Any odd member of M', say m_0', must be excluded from Construction C, implying that the even member $m_0'-1$ is excluded from both A and B. But we have seen that the only even number excluded from A and B is $m_e'=2$, so the only odd member of M' is $m_0'=3$. M' therefore contains only the two integers 2 and 3, and the theorem is established.

Reference

1. Claude Berge, The Theory of Graphs and its Applications, Wiley, New York, 1962, pp. 35–36.

A METHOD OF TRISECTION OF AN ANGLE AND X-SECTION OF AN ANGLE

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1. Parallel lines on a cylinder.

THEOREM I. *If a line on the surface of the cylinder is parallel to the center line, then the length of this line will not change if we rotate the cylinder and look at the side view.*

Proof. Draw $P_1P_1 \| P_2P_2$, $AB \perp P_1P_1$ and P_2P_2, $A'B' \perp P_1P_1$ and P_2P_2; then $AB = A'B'$ and $AB \| A'B'$. If we draw the same picture as above on a cylinder with both lines AB and $A'B'$ parallel to the center line CC' (Figure 2), the result will be the same: $AB = A'B'$ and $AB \| A'B'$. Suppose that we only draw the line AB on a cylinder and parallel to the center line CC' as in Figure 1 and then rotate the cylinder with the center line CC' as an axis. The line AB will then move up and down but its length never changes as we look at the side view (Figure 3).

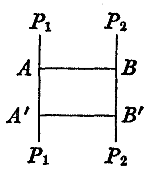

FIG. 1

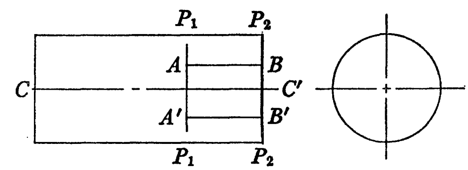

FIG. 2

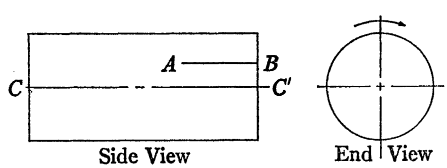

FIG. 3a

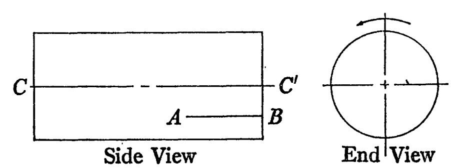

FIG. 3b

2. The relation between the cylinder and the sine curve.

THEOREM II. *If a straight line is set with an angle to the center line of a cylinder and this line is bent around the cylinder and kept in the same plane then the projection of this line on the center plane EFGH (Figure 4c) is a sine curve.*

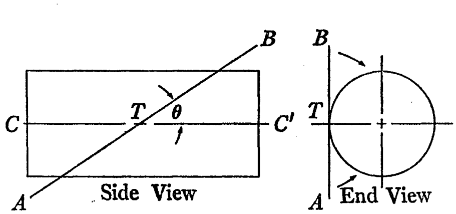

FIG. 4a

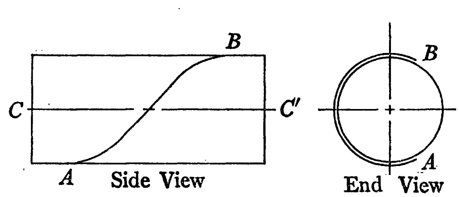

FIG. 4b

DEMONSTRATION A. Take a piece of string as AB tangent to the cylinder at point T with an angle θ to the center line CC' (Figure 4a). Then bend this string

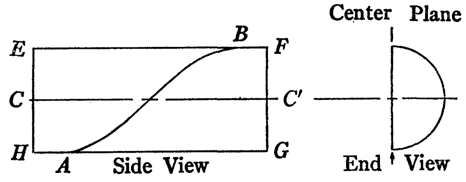

FIG. 4c

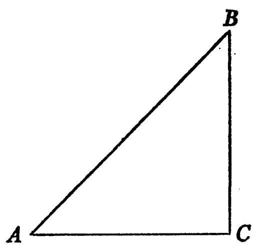

FIG. 5a

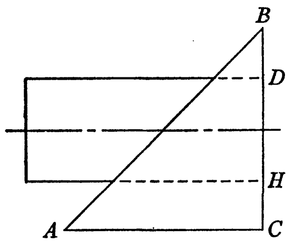

FIG. 5b

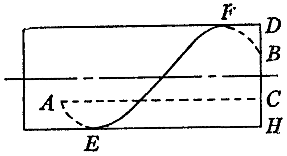

FIG. 5c

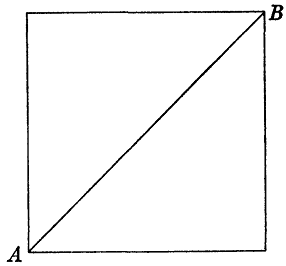

FIG. 6a

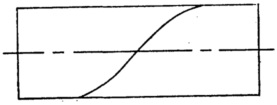

FIG. 6b

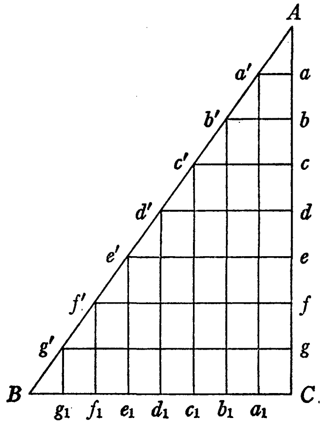

FIG. 7a

AB (as shown by the arrows in Figure 4a) with both ends A and B around the cylinder and keep the string AB in the same plane. Then when we look at the side view (Figure 4b) the string will have the semblance of a sine curve.

DEMONSTRATION B. Cut a piece of paper to make a right triangle ABC (Figure 5a). Then roll this triangle on a cylinder with the side BC along the edge DH of the cylinder (Figure 5b). Then when we look at the side view the line EF (a part of side AB) will have the semblance of a sine curve (Figure 5c).

DEMONSTRATION C. Cut a piece of soft cardboard about 6″ square. Draw a line AB diagonally on the cardboard (Figure 6a). Then roll up the cardboard as a cylinder (Figure 6b). The line AB will have the semblance of a sine curve.

DEMONSTRATION D. When you go to get a hair cut take a few minutes to watch the barber shop sign!

Proof. Cut a piece of paper to make a right triangle ABC (Figure 7a). Make side AC equal to a quarter of the circumference of a cylinder of diameter D. Divide the side AC of the right triangle ABC into a number of equal parts (say 8 parts): $Aa=ab=bc=cd=de=ef=fg=gC$. From each point a, b, c, d, e, f, and g draw lines parallel to BC and intercepting AB at a', b', c', d', e', f', and g' respectively. Then $Aa'=a'b'=b'c'=c'd'=d'e'=e'f'=f'g'=g'C$ and

$$\angle Aa'a = \angle a'b'b = \angle b'c'c = \angle c'd'd = \angle d'e'e = \angle e'f'f = \angle f'g'g = \angle g'BC.$$

From each point a', b', c', d', e', f' and g' draw a line perpendicular to BC; then $Ca_1=a_1b_1=b_1c_1=c_1d_1=d_1e_1=e_1f_1=f_1g_1=g_1B$.

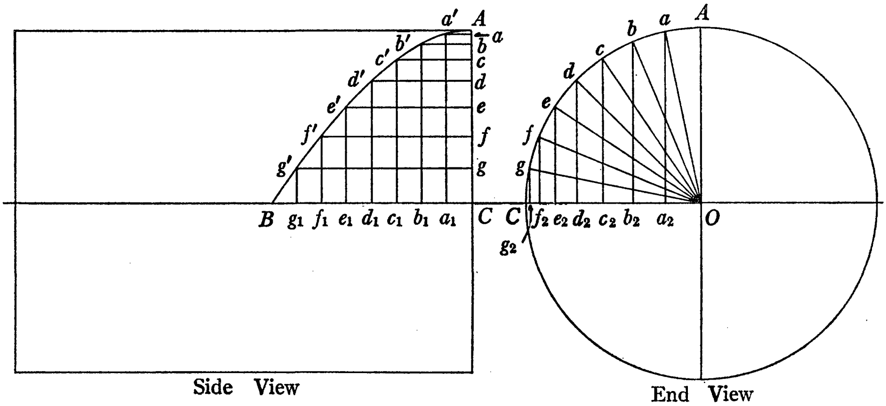

FIG. 7b

Paste the right triangle ABC on the cylinder and match the point A of the triangle with the point A on the top of the cylinder. Let the side AC of the triangle drop along the edge AC of the cylinder (Figure 7b). Then the lines $a'a$, $b'b$, $c'c$, $d'd$, $e'e$, $f'f$, and $g'g$ are all parallel to the center line of the cylinder. Mark the points a, b, c, d, e, f, and g on the end view of the cylinder also. Connect these points to the center O (Figure 7b). Then

$$\widehat{Aa} = \widehat{ab} = \widehat{bc} = \widehat{cd} = \widehat{de} = \widehat{ef} = \widehat{fg} = \widehat{gC}$$

and

$$\angle AOa = \angle aOb = \angle bOc = \angle cOd = \angle dOe = \angle eOf = \angle fOg = \angle gOC.$$

From each point (Figure 7b, end view) a, b, c, d, e, f, and g draw line segments aa_2, bb_2, cc_2, dd_2, ee_2, ff_2, and gg_2 perpendicular to the center line OC. It is clear the ratio of each line segment to the radius is the sine of the opposite angle, e.g.,

$$\frac{aa_2}{Oa} = \sin \angle aOa_2, \qquad \frac{bb_2}{Ob} = \sin \angle bOb_2, \text{ etc.}$$

Look at the side view of Figure 7b. The segments Ca_1, a_1b_1, b_1c_1, c_1d_1, d_1e_1, e_1f_1, f_1g_1, and g_1B always remain the same length (Theorem I) and the parallel line segments $a'a$, $b'b$, $c'c$, $d'd$, $e'e$, $f'f$, and $g'g$ also remain the same length (Theorem I). But the space between these parallel lines $a'a$, $b'b$, $c'c$, $d'd$, $e'e$, $f'f$, and $g'g$ changes. That is $Aa < ab < bc < cd < de < ef < fg < gC$ and so the angles also change in size. That is, $\angle Aa'a < \angle a'b'b < \angle b'c'c < \angle c'd'd < \angle d'e'e < \angle e'f'f < \angle f'g'g < \angle g'BC$. This means the tangent at each point is changed. So when we look at the side view, AB is a curve and not a straight line.

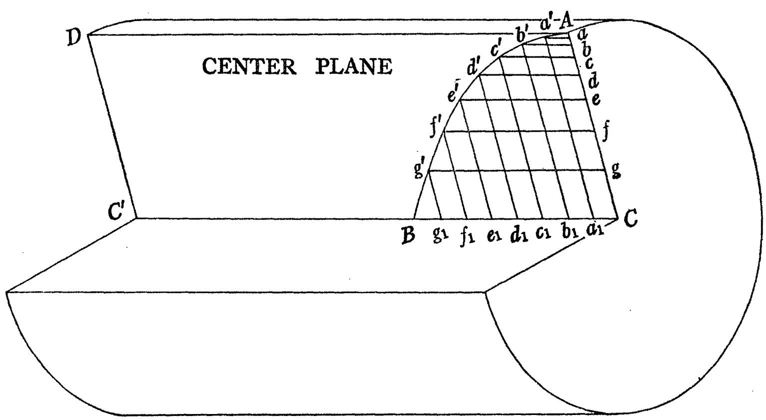

FIG. 8

Now if we project the side view of Figure 7b to the center plane $ACC'D$ of the cylinder (Figure 8 as shown in perspective view for easier observance), we will have the same picture on the side view as well as on the center plane. So the lines $a'a_1$, $b'b_1$, $c'c_1$, $d'd_1$, $e'e_1$, $f'f_1$, and $g'g_1$ are all straight and all perpendicular to the center line CC'; then $a'a_1 = aa_2$, $b'b_1 = bb_2$, $c'c_1 = cc_2$, $d'd_1 = dd_2$, $e'e_1 = ee_2$, $f'f_1 = ff_2$, and $g'g_1 = gg_2$. All of these results are fulfilled in a sine curve. So AB is a sine curve on the center plane.

Now we come back to the side view of Figure 7b. On the sine curve AB, $\widehat{c'd'}$ is not equal to $\widehat{d'e'}$; on AC, cd is not equal to de. In the end view, however, $\widehat{cd}$ is the arc of $\angle cOd$ and $\widehat{de}$ is the arc of $\angle dOe$ and these two angles are equal. On BC of the side view, c_1d_1 is equal to d_1e_1 and never changes. From this relationship, if we want to find two equal angles on the circle (end view of the cylinder), we must have the equal segments on the line BC (side view). Then (Figure 9a) from each point which cuts the equal segments on the line BC we draw lines perpendicular to line BC and intercepting the sine curve at certain points, say X_1, Y_1, and Z_1. Then from X_1, Y_1, and Z_1 draw lines parallel to the line BC and intercepting AC at the points X_2, Y_2, and Z_2. (This step may be omitted, because the parallel lines can intercept the circle directly.) Extend these parallel lines to intercept the circle at points X_3, Y_3, and Z_3. Connect X_3O, Y_3O and Z_3O; then $\angle X_3OY_3 = \angle Y_3OZ_3$.

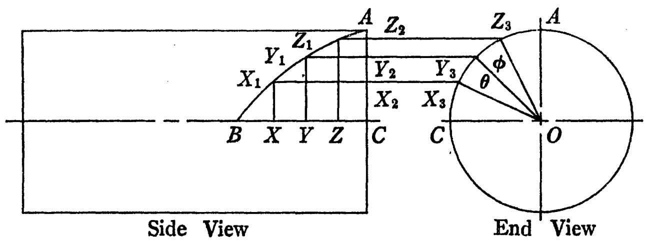

FIG. 9a

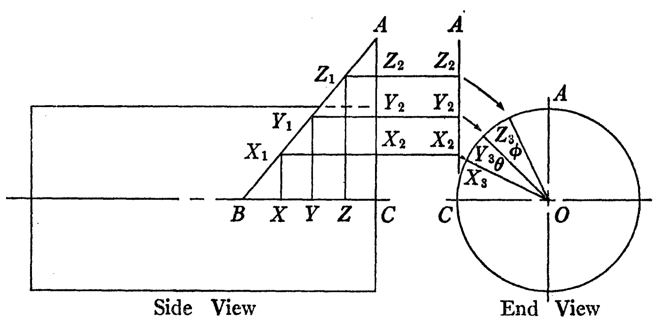

FIG. 9b

Suppose by imagination we lift the point A up as shown in Figure 9b; then ABC is a right triangle. It is clear that $XY = YZ$, $X_1Y_1 = Y_1Z_1$, and $X_2Y_2 = Y_2Z_2$ by construction. Suppose we push $\triangle ABC$ back to the cylinder. Then X_2 will drop on X_3, Y_2 will drop on Y_3, and Z_2 will drop on Z_3. So $\widehat{X_3Y_3} = \widehat{Y_3Z_3}$; that is, $\angle\theta = \angle\phi$. This means that $XY : YZ = \angle\theta : \angle\phi = 1:1$: Suppose $XY \neq YZ$; then $XY : YZ = X_1Y_1 : Y_1Z_1 = X_2Y_2 : Y_2Z_2 = \widehat{X_3Y_3} : \widehat{Y_3Z_3} = \angle\theta : \angle\phi$. The ratio never changes. It is possible to cut two segments $XY = YZ$ or cut two arcs $\widehat{XY} = \widehat{YZ}$ by using a compass. But it is impossible to cut a segment equal to an arc. So the sine curve acts as the intermediary in the relationship between segments and arcs.

THEOREM III. *If a point (point A in Figure 10 and Figure 11) or points (points A and C in Figure 12) on a circle divides an angle into two parts ($\angle\theta$ and $\angle\phi$ in Figure 10 and Figure 11) or more than two parts ($\angle\theta$, $\angle\phi$ and $\angle\psi$ in Figure 12), then from this point or points draw a line (AB in Figure 10 and Figure 11) or lines (AB and CD in Figure 12) parallel to the center line O_1O_2 of the circle* *and intercepting the sine curve at a certain point (point B in Figure* 10 *and Figure* 11) *or points (points B and D in Figure* 12). *From this point or points which are on the sine curve draw a line or lines perpendicular to the center line* O_1O_2 *and intercepting the center line at a certain point or points which divide the center line into different segments (a, b in Figure* 10 *and Figure* 11, *and a, b, and c in Figure* 12). *Then the ratios of these segments are in proportion to these divided angles (* $\angle\theta$, $\angle\phi$ *in Figure* 10 *and Figure* 11 *and* $\angle\theta$, $\angle\phi$, *and* $\angle\psi$ *in Figure* 12) *and vice versa.*

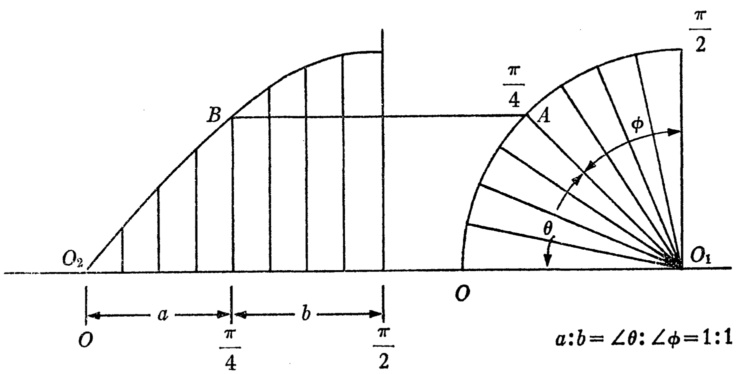

FIG. 10

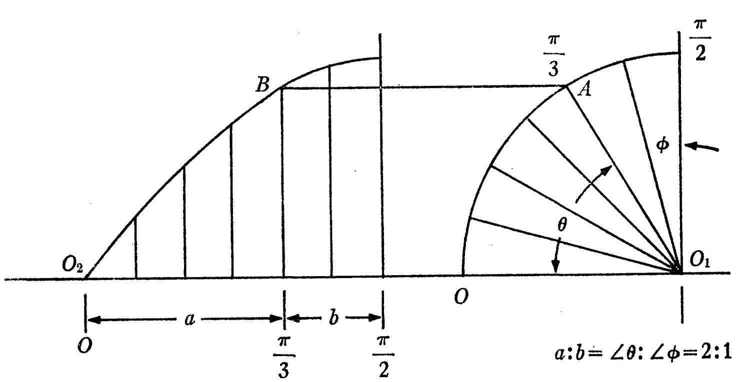

FIG. 11

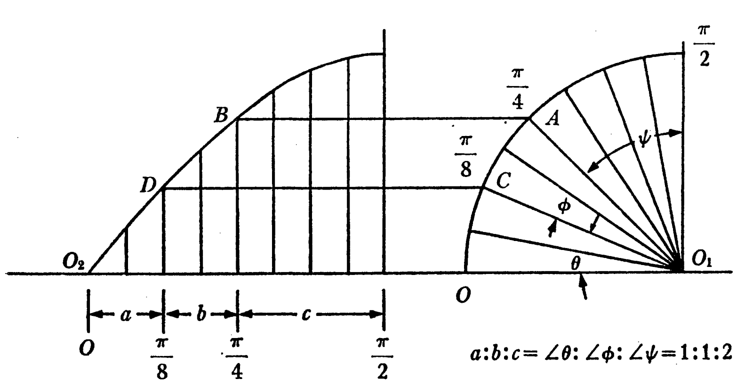

FIG. 12

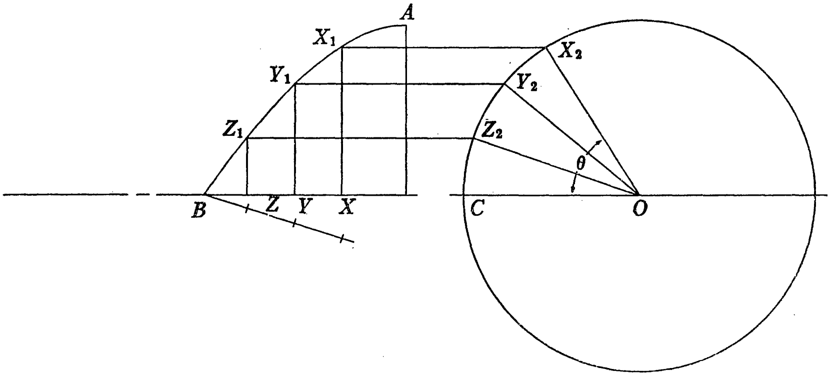

FIG. 13

3. **Trisection of an angle.** Let $\angle\theta$ be the given angle to be trisected (Figure 13). Taking O as the center, draw a circle intercepting both sides of $\angle\theta$ at X_2 and C. Draw the sine curve corresponding to the circle O. From X_2 draw a line parallel to the center line BC and intercepting the sine curve at X_1. From X_1 draw line X_1X perpendicular to BC and intercepting BC at X. Divide BX into 3 equal parts (theorem of parallel lines), $BZ=ZY=YX$. Draw lines Z_1Z and Y_1Y perpendicular to BC and intercepting the sine curve at Z_1 and Y_1. From points Z_1 and Y_1 draw Y_1Y_2 and Z_1Z_2 both parallel to X_1X_2 and intercepting the circle O at Y_2 and Z_2. Connect Y_2O and Z_2O. Then the lines Y_2O and Z_2O will divide $\angle\theta$ into 3 equal parts.

Proof. $BZ=ZY=YX$ by construction.

$$\angle X_2OY_2 = \angle Y_2OZ_2 = \angle Z_2OC \qquad \text{from Theorem III.}$$

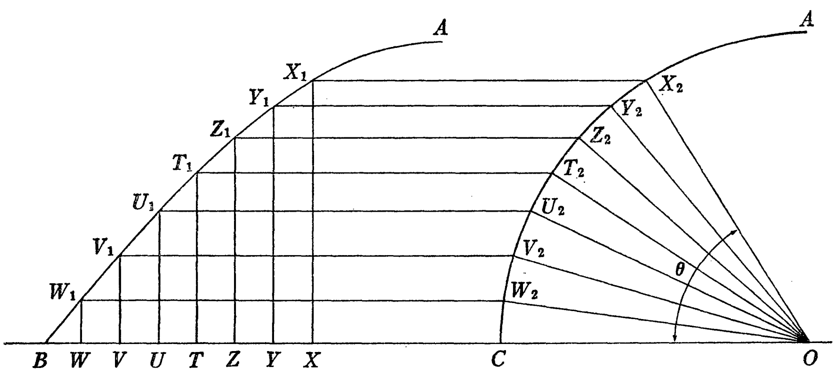

FIG. 14

4. ***X*-section of an angle, $X=7$.** Take the same steps as in the trisection of an angle but divide BX into 7 equal parts instead of 3 equal parts (Figure 14). Then $BW=WV=VU=UT=TZ=ZY=YX$. In a similar manner, find the points Y_1, Z_1, T_1, U_1, V_1, and W_1 on the sine curve and points Y_2, Z_2, T_2, U_2, V_2,

and W_2 on the circle O. Connect lines Y_2O, Z_2O, T_2O, U_2O, V_2O, and W_2O. Then these lines will divide $\angle\theta$ into 7 equal parts.

Proof. Similar to proof of theorem on trisection of an angle.

5. Compared with the method of Nicomedes.

A. This method takes fewer steps.

B. Constructing a sine curve is easier than constructing a conchoid curve. (Sine curve: $y=\sin x$; Conchoid of Nicomedes: $x^2y^2=(y+a)^2(b^2-y^2)$.)

C. This method can be used to divide a given angle into any number of equal parts.

D. Since there are an infinite number of points of the curve and only a finite number of them can be constructed, using the Conchoid curve for the trisection of an angle is not a purely geometric one.

SECTIONS OF n-DIMENSIONAL SPHERICAL CONES

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Many proofs have been given for the proposition: *The intersection of a right circular cone and a plane is a second degree curve (conic).* In this article, first, we supply a vector proof of this proposition. Then we generalize the theorem for sections of a spherical cone in an n-dimensional Euclidean space.

1. Notations. All spaces are real Euclidean. The inner product of two vectors will be denoted by (α, β). The norm of α is $\|\alpha\|$. The subspace spanned by the set of vectors $\{\alpha_1, \cdots, \alpha_k\}$ will be indicated by $[\alpha_1, \cdots, \alpha_k]$.

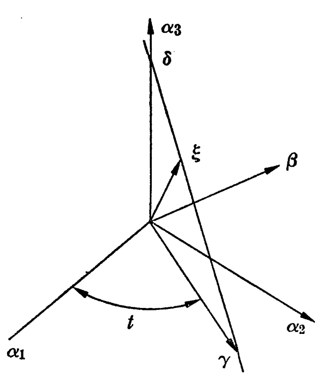

FIG. 1

2. THEOREM. *A plane and a right circular cone intersect in a second degree curve (conic section).*

Proof. Consider the orthonormal set $\{\alpha_1, \alpha_2, \alpha_3\}$ (Figure 1). Let the vector $\gamma\in[\alpha_1, \alpha_2]$ such that

$$\gamma = (a \cos t)\alpha_1 + (a \sin t)\alpha_2,$$

where a, a positive fixed real number, is the radius of the circle generated by γ. Choose the vector δ on α_3 such that $\delta = b\alpha_3$, where b is a fixed, real positive number. We can now consider the right circular cone generated by the line passing through the end points of γ and δ and about an axis whose unit vector is α_3.

Consider any vector ξ ending on the surface of the cone. It is clear that $\xi = p\gamma + q\delta$, $p + q = 1$.

Let β be a vector such that $\beta \in [\alpha_2, \alpha_3]$ and $\|\beta\| = 1$. Then $\beta = (\beta, \alpha_2)\alpha_2 + (\beta, \alpha_3)\alpha_3$. Consider the plane $[\alpha_1, \beta]$. Now in order to discuss the intersection of this plane and the cone we must also have $\xi \in [\alpha_1, \beta]$. Then $\xi = x\alpha_1 + y\beta$. Therefore $\xi = x\alpha_1 + y(\beta, \alpha_2)\alpha_2 + y(\beta, \alpha_3)\alpha_3$ for ξ in the plane, and $\xi = p(a \cos t)\alpha_1 + p(a \sin t)\alpha_2 + qb\alpha_3$ for ξ ending on the cone. These equalities imply that

$$\begin{aligned} x &= ap \cos t, \\ y(\beta, \alpha_2) &= ap \sin t, \\ y(\beta, \alpha_3) &= qb, \quad \text{where } p + q = 1. \end{aligned}$$

Suppose $\sin t$ and $\cos t$ are different from zero. Then

$$p = \frac{x}{a \cos t}, \qquad p = \frac{y(\beta, \alpha_2)}{a \sin t}, \quad \text{and} \quad q = \frac{y(\beta, \alpha_3)}{b}.$$

Therefore we have the equations

$$\begin{aligned} \frac{x}{a \cos t} + \frac{y(\beta, \alpha_3)}{b} &= 1, \\ \frac{y(\beta, \alpha_2)}{a \sin t} + \frac{y(\beta, \alpha_3)}{b} &= 1. \end{aligned}$$

This set of equations implies that

$$\frac{x}{a \cos t} = \frac{y(\beta, \alpha_2)}{a \sin t}.$$

Thus

$$\begin{aligned} \frac{a}{b} \cos t &= \frac{x}{b - y(\beta, \alpha_3)}, \\ \frac{a}{b} \sin t &= \frac{y(\beta, \alpha_2)}{b - y(\beta, \alpha_3)}, \qquad b - y(\beta, \alpha_3) \neq 0. \end{aligned}$$

This set of equations implies that

$$\frac{a^2}{b^2}(\cos^2 t + \sin^2 t) = \frac{x^2 + y^2(\beta, \alpha_2)^2}{[b - y(\beta, \alpha_3)]^2}.$$

Consequently $b^2x^2 + b^2(\beta, \alpha_2)^2y^2 = a^2[b - y(\beta, \alpha_3)]^2$ which proves the theorem.

Special cases where either cos t or sin t or $b-y(\beta, \alpha_3)$ is zero should be discussed. We omit the discussion.

3. Spherical cones. Let E_n be an n-dimensional Euclidean space. Choose the orthonormal basis $\{\alpha_1, \cdots, \alpha_n\}$ in E_n. Let $\gamma \in [\alpha_1, \cdots, \alpha_{n-1}]$ with $\|\gamma\| = r > 0$. Then there are real numbers a_i, $i=1, \cdots, n-1$ for which $\gamma = \sum_{i=1}^{n-1} a_i\alpha_i$ and $\sum_{i=1}^{n-1} a_i^2 = r^2$. Choose $\delta = b\alpha_n$, where b is a fixed, real positive number. Let $\xi = p\gamma + q\delta$, $p+q=1$. The locus of ξ (end point of ξ) as γ changes is called an *$(n-1)$-dimensional right spherical cone.*

4. THEOREM. *A hyperplane intersects a right spherical cone in a quadric.*

Proof. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the orthonormal set of Section 3. Choose the orthonormal set $\{\beta_2, \cdots, \beta_{n-1}\}$ such that $\beta_i \in [\alpha_2, \cdots, \alpha_n]$ for $i=2, \cdots, n-1$. Then

$$\beta_i = (\beta_i, \alpha_2)\alpha_2 + \cdots + (\beta_i, \alpha_n)\alpha_n,\ i=2, \cdots, n-1. \tag{1}$$

Consider the hyperplane $[\alpha_1, \beta_2, \cdots, \beta_{n-1}]$. Let ξ be in the intersection of this hyperplane and the cone. Then $\xi \in [\alpha_1, \beta_2, \cdots, \beta_{n-1}]$ and ξ satisfies the definition in **3.** Thus

$$\xi = x_1\alpha_1 + x_2\beta_2 + \cdots + x_{n-1}\beta_{n-1}$$
$$\xi = p(a_1\alpha_1 + \cdots + a_{n-1}\alpha_{n-1}) + qb\alpha_n, \qquad p+q=1.$$

However, by (1), we have

$$\begin{aligned}\xi &= x_1\alpha_1 + x_2[(\beta_2, \alpha_2)\alpha_2 + \cdots + (\beta_2, \alpha_n)\alpha_n] \\ &\quad + \cdots + x_{n-1}[(\beta_{n-1}, \alpha_2)\alpha_2 + (\beta_{n-1}, \alpha_n)\alpha_n] \\ &\equiv p(a_1\alpha_1 + \cdots + a_{n-1}\alpha_{n-1}) + qb\alpha_n, \qquad p+q=1.\end{aligned}$$

This identity implies that

$$\begin{aligned} x_1 &= pa_1 \\ x_2(\beta_2, \alpha_2) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_2) &= pa_2 \\ \cdots\cdots\cdots\cdots\cdots & \\ x_2(\beta_2, \alpha_{n-1}) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_{n-1}) &= pa_{n-1} \\ x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n) &= qb, \qquad p+q=1. \end{aligned}$$

Suppose $a_1, \cdots, a_{n-1}$ are all different from zero. Then

$$p = \frac{x_1}{a_1}$$

$$p = \frac{x_2(\beta_2, \alpha_2) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_2)}{a_2}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$p = \frac{x_2(\beta_2, \alpha_{n-1}) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_{n-1})}{a_{n-1}}$$

$$q = \frac{x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)}{b}, \qquad \text{where } p + q = 1.$$

Therefore we have the following equations

$$\frac{x_1}{a_1} + \frac{x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)}{b} = 1$$

$$\frac{x_2(\beta_2, \alpha_2) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_2)}{a_2} + \frac{x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)}{b} = 1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{x_2(\beta_2, \alpha_{n-1}) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_{n-1})}{a_{n-1}} + \frac{x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)}{b} = 1.$$

Thus

$$\frac{a_1}{b} = \frac{x_1}{b - [x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)]}$$

$$\frac{a_i}{b} = \frac{x_2(\beta_2, \alpha_i) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_i)}{b - [x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)]}, \qquad i = 2, \cdots, n-1.$$

This set of equations implies that $a_1^2/b^2 + \cdots + a_{n-1}^2/b^2$

$$= \frac{x_1^2 + [x_2(\beta_2, \alpha_2) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_2)]^2 + \cdots + [x_2(\beta_2, \alpha_{n-1}) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_{n-1})]^2}{\{b - [x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)]\}^2},$$

where $b - [x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)] \neq 0$ and $\sum_{i=1}^{n-1} a_i^2 = r^2$ as defined in Section **3**. Consequently

$$\begin{aligned} r^2\{b - [x_2(\beta_2, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)]\}^2 \\ = b^2\{x_1^2 + [x_2(\beta_2, \alpha_2) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_2)]^2 \\ + \cdots + [x_2(\beta_2, \alpha_{n-1}) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_{n-1})]^2\} \end{aligned}$$

which proves the theorem.

Special cases where $a_1, \cdots, a_{n-1}$ are zero or $b - [x_2(\beta_2, \alpha_n) + \cdots + x_{n-1} \cdot (\beta_{n-1}, \alpha_n)]$ is zero should be discussed. We omit the discussion.

5. Note. In Section 4 if we set $\beta_i = \alpha_i$, $i = 2, \cdots, k$, $k \leqq n-1$, then the hyperplane becomes $[\alpha_1, \cdots, \alpha_k, \beta_{k+1}, \cdots, \beta_{n-1}]$. Thus the equation of the intersection will be

$$\begin{aligned} r^2\{b - [x_{k+1}(\beta_{k+1}, \alpha_n) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_n)]\}^2 \\ = b^2\{x_1^2 + \cdots + x_k^2 \\ + [x_{k+1}(\beta_{k+1}, \alpha_{k+1}) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_{k+1})]^2 \\ + \cdots + [x_{k+1}(\beta_{k+1}, \alpha_{n-1}) + \cdots + x_{n-1}(\beta_{n-1}, \alpha_{n-1})]^2\}. \end{aligned}$$

In this case the $(n-1)$-dimensional quadric has spherical sections.

ON THE MINIMAL RECTANGULAR REGION WHICH HAS THE LATTICE POINT COVERING PROPERTY

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1. Introduction. A set, S, of points in the Euclidean plane is said to have the *lattice point covering property* [2] if and only if every set obtained by a rigid motion of S always contains a lattice point, that is, a point whose coordinates are integers. Necessary and sufficient conditions that a rectangular region of dimension a by b where $a \leqq b$ have the lattice point covering property are that $a \geqq 1$ and $b \geqq \sqrt{2}$ (see [1]). It is the purpose of this note to give an alternative proof of the sufficiency of these conditions.

2. Notation. Throughout the following E will denote the Euclidean plane, U_a will denote the half-open square of side a given by $\{(x, y) \mid 0 \leqq x < a, 0 \leqq y < a\}$, and R will denote rectangular region given by $\{(x, y) \mid 0 \leqq x \leqq \sqrt{2}, 0 \leqq y \leqq 1\}$. If α and β are points in E, $\alpha + \beta$ will denote their vector sum. If $\alpha \in E$ and $S \subset E$, $S + \alpha$ will denote $\{\beta + \alpha \mid \beta \in S\}$. If α has coordinates (x, y), $\langle \alpha \rangle$ will denote the point with coordinates $(x - [x], y - [y])$, i.e., the point whose coordinates are the fractional parts of the coordinates of α, and $\langle S \rangle$ will denote $\{\langle \alpha \rangle \mid \alpha \in S\}$. Finally, θS will denote the set obtained by rotating S through an angle θ in a counterclockwise direction about the origin. In this notation, what we wish to prove is that $\theta R + \alpha$ contains a lattice point for all $\alpha \in E$ and $0 \leqq \theta < \pi$.

3. Proof of the theorem. Lemma 1 is essentially a lemma of Sawyer's (see [3]). The lemma gives necessary and sufficient conditions that every translation of a set always contains a lattice point.

LEMMA 1. *Let $S \subset E$; then $S + \alpha$ contains a lattice point for every $\alpha \in E$ if and only if $\langle S \rangle = U_1$.*

Proof. Suppose for every $\alpha \in E$, $S + \alpha$ contains a lattice point. Let $\beta \in U_1$. Then $S - \beta$ contains a lattice point, say τ. Hence there is a point γ of S such that $\gamma = \tau + \beta$. Thus $\beta = \langle \gamma \rangle$ and therefore $U_1 \subset \langle S \rangle$. Finally $\langle S \rangle \subset U_1$ by the definition of $\langle S \rangle$.

Conversely, suppose $\langle S \rangle = U_1$. Let $\alpha \in E$; then $\langle -\alpha \rangle \in U_1 = \langle S \rangle$. Therefore there exists $\gamma \in S$ such that $\langle \gamma \rangle = \langle -\alpha \rangle$. Hence there is a lattice point τ such that $\gamma = -\alpha + \tau$. Thus $\tau = \gamma + \alpha \in S + \alpha$.

LEMMA 2. *Let $0 \leqq \theta \leqq (\pi/2)$ and let*

$$D_\theta = U_{\cos\theta} \cup (U_{\sin\theta} + (\cos\theta, \cos\theta - \sin\theta))$$

(see Figure 1); *then for any $\alpha \in E$, $\theta D_\theta + \alpha$ always contains a lattice point.*

Proof. Let $A = \{(x, y) \mid x < 0\}$ and $B = \{(x, y) \mid y \geqq 1\}$. Let $T_1 = \theta D_\theta \cap A$ and $T_2 = \theta D_\theta \cap B$. Then (see Figure 2),

$$\langle \theta D_\theta \rangle = (\theta D_\theta - (T_1 \cup T_2)) \cup (T_1 + (1, 0)) \cup (T_2 + (0, -1)) = U_1$$

and the result follows by Lemma 1.

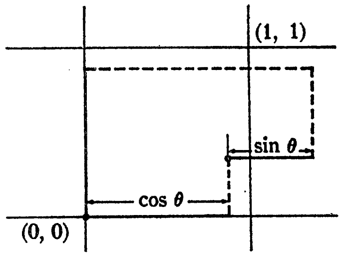

FIG. 1. D_θ.

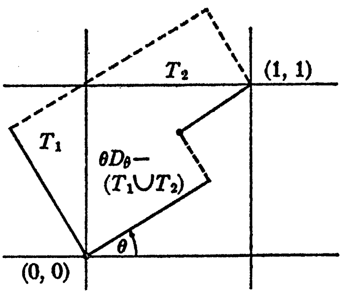

FIG. 2. θD_θ.

It is interesting to note that θD_θ has area 1 which, by Lemma 1, is the minimal area a set may have and still have every translation of the set contain a lattice point.

THEOREM. *If* $0 \leqq \theta < \pi$ *and* $\alpha \in E$, *then* $\theta R + \alpha$ *contains a lattice point.*

Proof. Case 1. $0 \leqq \theta \leqq \pi/2$. Since for $0 \leqq \theta \leqq \pi/2$, $\sin \theta \leqq 1$, $\cos \theta \leqq 1$ and $\sin \theta + \cos \theta \leqq \sqrt{2}$, it follows that $D_\theta \subset R$. Then $\theta D_\theta + \alpha \subset \theta R + \alpha$ and the result follows by Lemma 2.

Case 2. $\pi/2 < \theta < \pi$. In this case $(\theta - (\pi/2))R + \alpha$ contains a lattice point, (m, n), by Case 1. Then $\theta R + \alpha$ contains the lattice point $(-n, m)$.

References

1. Ivan Niven, Diophantine Approximations, Interscience, New York, 1963, p. 54.

2. ——— and H. S. Zuckerman, Lattice point coverings by plane figures, Amer. Math. Monthly, 74 (1967) 353–362.

3. D. B. Sawyer, On the covering of lattice points by convex regions, Quart. J. Math., (1953) 286.

THE LATTICE POINT COVERING THEOREM FOR RECTANGLES

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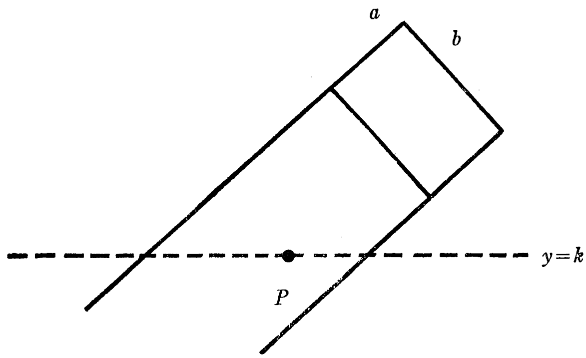
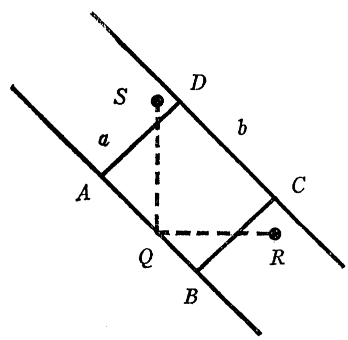
Say that a rectangle has the lattice point covering property if it contains a point with integral coordinates in its interior or on its boundary when set down on the coordinate plane, regardless of position or orientation. It is known [1] that a rectangle with sides a, b with $a \leqq b$ has the lattice point covering property if and only if $a \geqq 1$ and $b \geqq \sqrt{2}$. Another proof has been given by E. A. Maier [2] in the preceding paper in this issue. We give here a third and different proof.

First we set aside the rectangles with $a < 1$, because any such rectangle can be placed between the lines $y = 0$ and $y = 1$, with the sides of length b parallel to these lines. Thus any rectangle with $a < 1$ does not have the lattice point covering property.





Henceforth we consider only rectangles with $a \geqq 1$. If $b < \sqrt{2}$ the rectangle can be placed with center at $(\frac{1}{2}, \frac{1}{2})$ and sides parallel to $y=x$ and $y=-x$, and in this position the rectangle contains no lattice points.

Conversely consider a rectangle with $a \geqq 1$ lacking the lattice point covering property. With the rectangle placed on the plane so as to contain no lattice point, extend the sides of length a to form a strip. Some line $y=k$ or $x=k$, with k an integer, will cross the strip. The part of this line that lies in the strip has length at least $b \geqq a \geqq 1$. Therefore there is a lattice point P in the strip. Slide the rectangle down the strip until it first touches a lattice point. Thus a lattice point Q now lies on the side AB, with AB of length b. There may be other lattice points on the side AB, but there are no others covered by the rectangle. Draw the coordinate lines through Q. Since $AD=a \geqq 1$ each of these coordinate lines will contain a lattice point one unit from Q in the strip formed by AB, CD and their extensions. Call these lattice points R and S. If R were on AB then S would be covered by the rectangle, but it would not be on AB. Thus R is not on AB; and similarly S is not on AB. Therefore R and S lie outside the rectangle, but they lie in the strip formed by the extensions of AB and CD. Clearly one of R and S lies in the part of the strip beyond BC, while the other lies in the part beyond AD. The distance between R and S is $\sqrt{2}$ and hence $b=AB<RS=\sqrt{2}$.

References

1. Ivan Niven, Diophantine Approximations, Interscience, New York, 1963, pp. 54–56.

2. E. A. Maier, On the minimal rectangular region which has the lattice point covering property, (see preceding article in this MAGAZINE).

MORLEY'S TRIANGLE

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Adjacent trisectors of the angles of any triangle ABC intersect (as shown in Figure 1) to form a smaller triangle PQR, which has come to be known as Morley's Triangle. It is Morley's Theorem that triangle PQR is equilateral, regardless of the shape of triangle ABC.

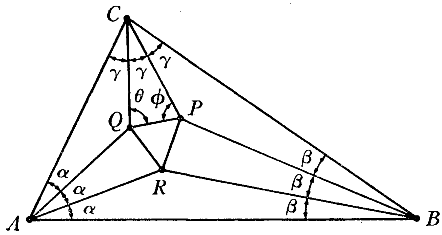

FIG. 1

As far as the authors are aware, Morley gave his theorem without proof at the close of the nineteenth century, and it was about thirty years before someone found a way to prove it. Here a direct proof of Morley's Theorem is given.

As a preliminary, we establish the following identity, valid for any angle ω:

$$
\begin{aligned}
\sin 3\omega &= \sin \omega(3 \cos^2 \omega - \sin^2 \omega) \\
&= 4 \sin \omega\, (\sin^2 60° \cos^2 \omega - \cos^2 60° \sin^2 \omega) \\
&= 4 \sin \omega \sin (60° + \omega) \sin (60° - \omega) \\
&= 4 \sin \omega \sin (60° + \omega) \sin (120° + \omega)
\end{aligned} \tag{1}
$$

From the sum of the angles in triangle ABC,

$$\alpha + \beta + \gamma = 60° \tag{2}$$

From the sum of the angles in each of triangles BCP, ACQ, and CPQ, and using Equation (2) to eliminate γ from each such sum, we have, respectively,

$$\begin{Bmatrix} \angle BPC = 120° + \alpha \\ \angle CQA = 120° + \beta \end{Bmatrix} \tag{3}$$

$$\phi + \theta = 120° + \alpha + \beta \tag{4}$$

By the law of sines for triangle ABC,

$$\overline{BC}/ \sin 3\alpha = \overline{AC}/\sin 3\beta = D \tag{5}$$

where D is a constant (D is in fact the diameter of the circumscribed circle of triangle ABC). Again, by the law of sines for triangles BCP, ACQ, and CPQ, respectively,

$$\overline{PC} = \overline{BC} \sin \beta / \sin \angle BPC \tag{6}$$
$$\overline{QC} = \overline{AC} \sin \alpha / \sin \angle CQA$$

$$\sin \theta / \sin \phi = \overline{PC}/\overline{QC}. \tag{7}$$

Upon successive substitution from Equations (5), (1), and (3), Equations (6) become

$$\overline{PC} = 4D \sin \beta \sin \alpha \sin (60^\circ + \alpha) \tag{8}$$
$$\overline{QC} = 4D \sin \alpha \sin \beta \sin (60^\circ + \beta).$$

Substitution of Equations (8) into Equation (7) then yields

$$\sin \theta / \sin \phi = \sin (60^\circ + \alpha)/\sin (60^\circ + \beta). \tag{9}$$

Now the two unknowns (ϕ, θ) are determined by the simultaneous Equations (4) and (9). By inspection it is seen that these equations are satisfied by

$$\theta = 60^\circ + \alpha \tag{10}$$
$$\phi = 60^\circ + \beta.$$

This is the unique solution within the meaningful ranges for ϕ and θ because Equations (4) and (9) determine the two remaining angles of a triangle CPQ for which two sides and their included angle $(\overline{PC}, \gamma, \overline{QC})$ are given; the side-angle-side congruency theorem of Euclid precludes the existence of more than one solution.

Similar analysis of the remaining triangles will complete the derivation of the following ninefold set of equations:

$$\angle RPB = 60^\circ + \gamma, \ \angle BPC = 120^\circ + \alpha, \ \angle CPQ = 60^\circ + \beta \tag{11}$$

$$\angle PQC = 60^\circ + \alpha, \ \angle CQA = 120^\circ + \beta, \ \angle AQR = 60^\circ + \gamma \tag{12}$$

$$\angle QRA = 60^\circ + \beta, \ \angle ARB = 120^\circ + \gamma, \ \angle BRP = 60^\circ + \alpha. \tag{13}$$

Finally, the sum of the angles surrounding each of the points P, Q, and R must be 360°; therefore we have from Equations (11), (12), and (13) with the aid of Equation (2) that $\angle RPQ = \angle PQR = \angle QRP = 60^\circ$ which completes the proof.

As a corollary, we have $\overline{QR} = \overline{RP} = \overline{PQ} = 4D \sin \alpha \sin \beta \sin \gamma$ which follows from Equations (8) and (10) and the law of sines for triangle CPQ.

A NEW APPROACH TO AN OLD PROBLEM

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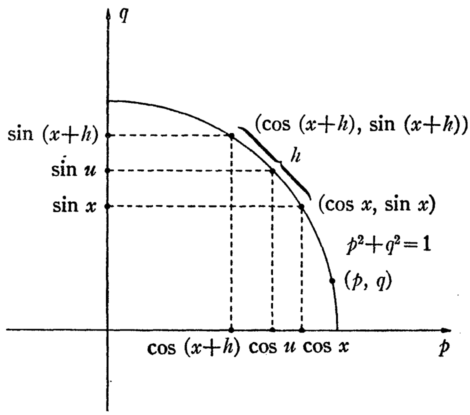
A common proof of $D_x(\sin x) = \cos x$ requires the use of the identity $\sin (x+h) = \sin x \cos h + \cos x \sin h$ and the establishment of $\lim_{h\to 0} (\sin h)/h = 1$ and $\lim_{h \to 0} (\cos h - 1)/h = 0$. The following proof avoids these ideas by making use of the formula for finding arc length and the mean value theorem for the definite integral.



THEOREM 1. $D_x(\sin x) = \cos x$.

Proof. For simplicity, assume that $(\cos(x+h), \sin(x+h))$ and $(\cos x, \sin x)$ are two points in the first quadrant on the circle $p^2+q^2=1$ where h is the arc length between these two points. The formula for finding arc length h yields

$$h = \int_{\sin x}^{\sin(x+h)} \sqrt{1 + [D_q(1-q^2)^{\frac{1}{2}}]^2}\, dq$$

$$= \int_{\sin x}^{\sin(x+h)} \frac{1}{\sqrt{1-q^2}}\, dq.$$

By the mean value theorem for the definite integral we obtain

$$h = \frac{1}{\sqrt{1-\sin^2 u}} (\sin(x+h) - \sin x),$$

where $\sin x \leqq \sin u \leqq \sin(x+h)$, since $x \leqq u \leqq (x+h)$ and the sine function is monotone increasing in the first quadrant. Then $\cos u = \sqrt{\cos^2 u} = (\sin(x+h) - \sin x)/h$ since $\cos u > 0$. As $h \to 0$, then $\cos(x+h) \to \cos x$ and $\cos u \to \cos x$ since $x \leqq u \leqq (x+h)$, so that

$$\cos x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = D_x(\sin x).$$

The same type of proof as above shows that $D_x(\cos x) = -\sin x$.

THEOREM 2. $D_x(\cos x) = -\sin x$.

Proof. Under the same assumptions as in Theorem 1, the formula for finding arc length h yields

$$h = \int_{\cos(x+h)}^{\cos x} \sqrt{1 + [D_p(1-p^2)^{\frac{1}{2}}]^2}\, dp$$

$$= \int_{\cos(x+h)}^{\cos x} \frac{1}{\sqrt{1-p^2}}\, dp.$$

By the mean value theorem for the definite integral we obtain

$$h = \frac{1}{\sqrt{1 - \cos^2 u}} (\cos x - \cos (x + h)),$$

where $\cos(x+h) \leqq \cos u \leqq \cos x$, since $x \leqq u \leqq (x+h)$ and the cosine function is monotone decreasing in the first quadrant. Then $-\sin u = -\sqrt{\sin^2 u} = (\cos(x+h) - \cos x)/h$ since $\sin u > 0$. As $h \to 0$, then $\sin(x+h) \to \sin x$ and $\sin u \to \sin x$ since $x \leqq u \leqq (x+h)$, so that

$$- \sin x = \lim_{h \to 0} \frac{\cos (x + h) - \cos x}{h} = D_x (\cos x).$$

For both Theorems 1 and 2 above, it is assumed that the sine and cosine functions are continuous functions.

The author wishes to thank the reviewer for his valuable suggestions.

ANSWERS

A449. The sequences consisting of n zeros and ones may be separated into sets such that each set contains sequences which begin with one of the following blocks: 1, 01, 0001, 000001, $\cdots$, or all zeros if n is odd. It follows that

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) + F(n-4) + F(n-6) + \cdots \\ &= F(n-1) + F(n-2) - F(n-3) + F(n-3) + F(n-4) \\ &\quad + F(n-6) + \cdots \\ &= F(n-1) + 2F(n-2) - F(n-3). \end{aligned}$$

A450. If we put $z = -4 + 1/x$ where $x \neq 0$ we obtain the result from the fact that $\sqrt{z^2+8}$ is rational.

A451. f' has a removable discontinuity at t if and only if $\lim_{x \to t} f'(x)$ exists and is different from $f'(t)$. This never happens for $f'(t)$ is defined as $\lim_{x \to t}(f(x) - f(t)) / (x-t)$ and we apply l'Hospital's rule in evaluating this limit, thus obtaining $f'(t) = \lim_{x \to t} f'(x)$.

A452. Proceeding clockwise: 8×2, 1×9, 10×5, 3×6 surround 4×7. This solution is not unique.

A453. Evidently $p+1$ is divisible by 2. Since every finite field has order q^n where q is prime and n is a positive integer, the order of F must be 2^n, that is, $p+1 = 2^n$. Hence $p = 2^n - 1$ is a Mersenne prime.

(Quickies on page 104)

DIMENSION UNDER ANALYTIC MAPS

G. P. SPECK, Bradley University

After viewing the many elementary conformal transformations normally encountered in an introductory complex variables course, a student is apt to feel that a nonconstant analytic map carries a "curve" onto a "curve" and a "region" onto a "region". The construction of a continuous space-filling curve will usually prompt one to question his intuitive feel for preservation of "dimension" under any specific type of transformation, including the analytic ones.

The following is an attempt at a relatively elementary discussion of dimension under analytic maps. Virtually no knowledge of general dimension theory is assumed and, as will be seen, very little is needed to accomplish our objective.

Theorem 4 of this paper establishes that an analytic function maps the image of a continuously differentiable curve onto a set of dimension less than two; Theorem 6 shows that an analytic function with nonzero derivative maps a one dimensional image of a continuous curve onto a set of dimension one.

An example at the end of the paper shows that the dimension of a set need not be preserved under a nonconstant analytic map.

DEFINITION. *If S is a nonnull subset of E^2, then S has dimension* 2 (dim $S=2$) *iff S contains a nonnull subset of points open in E^2. S has dimension* 1 (dim $S=1$) *iff S contains a subset of points homeomorphic to the open interval* (0, 1), *but S does not contain a nonnull subset of points open in E^2. S has dimension* 0 (dim $S=0$) *iff S does not contain a subset of points homeomorphic to the open interval* (0, 1).

From this definition it is immediate that each nonnull subset of the plane is assigned a unique integer 0, 1, or 2.

THEOREM 1. *If F is a nonconstant analytic function defined on S where* dim $S=2$, *then* dim $F(S)=2$.

Proof. A nonconstant analytic function maps open sets onto open sets.

THEOREM 2. *If*

$$C: \begin{matrix} x = f(t) \\ y = g(t) \end{matrix} \quad a \leqq t \leqq b$$

and

$$|f(t_1) - f(t_2)| < K_1 |t_1 - t_2|^c, \quad |g(t_1) - g(t_2)| < K_2 |t_1 - t_2|^d$$

for all $t_1, t_2 \in [a, b]$ where K_1, K_2, c, and d are each greater than zero with $c+d>1$, then dim $C([a, b]) \neq 2$.

Proof. Consider the closed intervals $[p_{i-1}, p_i]$ where $p_i = a + (i(b-a))/n$, n is a positive integer, and $i = 1, 2, 3, \cdots, n$. $C([p_{i-1}, p_i])$ is contained in a rectangle of points which is $K_1(b-a)^c/n^c$ by $K_2(b-a)^d/n^d$ on a side since

$$|f(t_1) - f(t_2)| < K_1(b - a)^c/n^c, \quad |g(t_1) - g(t_2)| < K_2(b - a)^d/n^d$$

for every $t_1, t_2 \in [p_{i-1}, p_i]$. Thus $C([a, b])$ is contained in a union R of rectangles, where the area of R is less than $K_1K_2(b-a)^{c+d}/n^{c+d-1}$. Now if p is any point in $C([a, b])$ and Δ is any disk with center at p, then Δ cannot be contained in $C([a, b])$ since n can be chosen large enough to make $K_1K_2(b-a)^{c+d}/n^{c+d-1}$ less than the area of Δ. Hence dim $C([a, b])\neq 2$.

THEOREM 3. *If a continuous curve* $C: {x=f(t) \atop y=g(t)}\ a\leqq t\leqq b$ *has finite length (rectifiable), then* dim $C([a, b])\neq 2$.

Proof. Let $S=\{(x, y)\mid 0\leqq x\leqq 1,\ 0\leqq y\leqq 1\}$ and suppose $S\subset C([a, b])$. Consider $P_m=P_{in+j}=(i/(n-1),\ (j-1)/(n-1))\in S$ where $i=0, 1, 2, \cdots, n-1$; $j=1, 2, 3, \cdots, n$; $m=1, 2, 3, \cdots, n^2$. If $t_m\in C^{-1}(P_m)\subset[a, b]$, then $a=r_0\leqq r_1 < r_2 < \cdots < r_{n^2}\leqq r_{n^2+1}=b$ where $r_1, r_2, \cdots, r_{n^2}$ is the appropriate permutation of the numbers $t_1, t_2, \cdots, t_{n^2}$. Observe that

$$\sum_{i=1}^{n^2+1} \mid C(r_i) - C(r_{i-1}) \mid \geqq (n^2 - 1)/(n - 1) = n + 1$$

since $| C(r_i)-C(r_{i-1})| \geqq 1/(n-1)$ for $i=2, 3, \cdots, n^2$. Therefore the length of C is infinite, a contradiction, and S is not contained in $C([a, b])$.

In a similar manner it can be shown that no square of points can be contained in $C([a, b])$. Hence dim $C([a, b])\neq 2$.

THEOREM 4. *If the curve* $C: {x=f(t) \atop y=g(t)}\ a\leqq t\leqq b$ *has* $f'(t)$ *and* $g'(t)$ *continuous over* $[a, b]$, *if* $C([a, b])\subset D$ *where* D *is open, and if* F *is a function analytic in* D, *then* dim $FC([a, b])\neq 2$.

Proof. Let $F(z)=u(x, y)+iv(x, y)$ and $FC: {u=r(t) \atop v=s(t)}\ a\leqq t\leqq b$.

Then $r(t)=u(f(t), g(t))$, $s(t)=v(f(t), g(t))$, and, by the chain rule, $r'(t) =\partial u/\partial x\cdot dx/dt+\partial u/\partial y\cdot dy/dt$ and $s'(t)=\partial v/\partial x\cdot dx/dt+\partial v/\partial y\cdot dy/dt$. Hence $r'(t)$ and $s'(t)$ are continuous over $[a, b]$ which implies that the length of FC is finite, which in turn implies by Theorem 3 that dim $FC([a, b])\neq 2$. Alternatively, $r'(t)$ and $s'(t)$ continuous over $[a, b]$ implies that $r(t)$ and $s(t)$ each satisfies the Lipschitz condition of order one, and thus by Theorem 2 it follows that dim $FC([a, b])\neq 2$.

THEOREM 5. *If* A *and* B *are nonnull sets in* E^2, *if* B *is closed in* E^2, *and if neither* A *nor* B *has dimension* 2, *then* dim $(A\cup B)\neq 2$.

Proof. To prove that $A\cup B$ does not contain a set open in E^2, let us suppose that such a set S exists. Consider $p\in S$ and suppose $p\in A$. If any neighborhood of p contains points of A only, we have a contradiction on dim $A\neq 2$. Thus any neighborhood of p contains points of B. Hence $p\in B$ since B is closed. It follows that $p\in S$ implies $p\in B$, which is a contradiction on dim $B\neq 2$. (Note: It follows easily that if dim $A=1$ or dim $B=1$, then dim$(A\cup B)=1$. Also, if dim $A=0$ and dim $B=0$, then dim$(A\cup B)=0$.)

THEOREM 6. *If the continuous curve* $C: {x=f(t) \atop y=g(t)}\ a\leqq t\leqq b$ *has* dim $C([a, b])=1$, *if* $C([a, b])\subset D$ *where* D *is open, and if* F *is a given function analytic in* D *with* $F'(z)\neq 0$ *for every* $z\in D$, *then* dim $FC([a, b])=1$.

Proof. Certainly $FC([a, b])$ contains a set homeomorphic to the open interval

(0, 1) since $C([a, b])$ contains such a set and it is known that F is locally a homeomorphism. Thus dim $FC([a, b]) \geqq 1$.

Now consider $p \in C([a, b])$; since F is locally a homeomorphism there exists $S(p, \epsilon_p)$, a spherical neighborhood with center at p and radius ϵ_p, such that F is a homeomorphism on $\overline{S(p, \epsilon_p)}$, the closure of $S(p, \epsilon_p)$. The collection of all such spherical neighborhoods, $\{S(p, \epsilon_p)\}$, $p \in C([a, b])$, is an open cover of $C([a, b])$, which is compact since compactness is a continuous invariant. Thus a finite number of sets $\{S(p, \epsilon_i)\}_{i=1}^{k}$ from $\{S(p, \epsilon_p)\}$ covers $C([a, b])$. If we let $S_i = \overline{S(p, \epsilon_i)} \cap C([a, b])$, then S_i is closed in E^2 and dim $S_i \leqq 1$, where at least one S_i has dimension equal to one by the note on Theorem 5. That dim $F(S_i) \leqq 1$ follows from the fact that dimension is a homeomorphic invariant, where, as before, at least one $F(S_i)$ has dimension equal to one. $F(S_i)$ is compact since F is continuous and, therefore, $F(S_i)$ is closed by the Heine-Borel theorem. Finally, since $FC([a, b]) = \bigcup_{i=1}^{k} F(S_i)$, Theorem 5 gives dim $FC([a, b]) \leqq 1$, which coupled with our initial inequality yields dim $FC([a, b]) = 1$.

Example. If:

$A = \{re^{i\theta} \mid r \geqq 0, r \text{ is algebraic}, 0 \leqq \theta < \pi/2, \theta \text{ is algebraic}\}$,

$B = \{re^{i(\theta + (\pi/2))} \mid r > 0, r \text{ is algebraic}, \pi/2 \leqq \theta < \pi, \theta \text{ is transcendental}\}$,

$C = \{re^{i(\theta+\pi)} \mid r > 0, r \text{ is transcendental}, \pi \leqq \theta < 3\pi/2, \theta \text{ is algebraic}\}$,

$D = \{re^{i(\theta + (3\pi/2))} \mid r > 0, r \text{ is transcendental}, 3\pi/2 \leqq \theta < 2\pi, \theta \text{ is transcendental}\}$,

$E = \{re^{i\theta} \mid r > 0, r \text{ is transcendental}, \theta = 0\}$,

$F = \{re^{i\theta} \mid r \geqq 0, r \text{ is algebraic}, \theta = \pi/2\}$,

$R = A \cup B \cup C \cup D, S = R \cup E, T = E \cup F; f(z) = z^4$.

then:

$$\dim R = 0 \qquad \dim f(R) = 2$$
$$\dim S = 1 \qquad \dim f(S) = 2$$
$$\dim T = 0 \qquad \dim f(T) = 1.$$

THE APPLICATION OF A FUNCTION TO UNIONS AND INTERSECTIONS OF SETS

BYRON L. McALLISTER, Montana State University

Everybody knows that if X and Y are sets, f is a function (possibly multivalued) on a subset of X into Y, and C is a nonempty collection of subsets of X, then

$$f[\cup_{S \in C} S] = \cup_{S \in C} f[S] \tag{1}$$

but

$$f[\cap_{S \in C} S] \neq \cap_{S \in C} f[S]. \tag{2}$$

When I was younger I fancied that this fact violated the duality between unions and intersections to which I had become happily accustomed. So I asked everybody to "explain" the matter. But nobody seemed to know a better answer than that it is simply "what Mother Nature gives us."

Recently I ran across a better explanation. I can hardly be the first to have noticed it, but if I am not the last to have noticed it either, there may be readers who'd enjoy sharing the secret.

The fact is, of course, that duality is *not* violated. The definition of $f[S]$ is one of two natural, dual, definitions. Recall that

$$f[A] = \{y \mid (\exists x \in X)((x, y) \in f \text{ and } x \in A)\}.$$

Naturally one should then also define

$$f\{A\} = \{y \mid (\forall x \in X)((x, y) \in f \Rightarrow x \in A)\}.$$

We might say that $f[A]$ contains those elements of Y that have any preimage in A and that $f\{A\}$ contains those elements of Y that have all preimages in A. To make the duality between $f[A]$ and $f\{A\}$ perfectly clear, note that $f\{A\} = Y-f[X-A]$.

Replacing cups by caps and vice versa and brackets by braces we obtain the easily proved duals of (1) and (2). In fact, just as (2) can be strengthened to $f[\bigcup_{S\in C}S]\subseteq\bigcap_{S\in C}f[S]$, so also may its dual be strengthened to $f\{\bigcup_{S\in C}S\}\supseteq\bigcup_{S\in C}f\{S\}$.

(It should be remarked that our dual definition is not entirely unknown in the literature. See for example the *upper inverse* of Berge's *Topological Spaces*, Macmillan, 1963, p. 24.)

THE SEQUENCE $\{\sin n\}$

C. STANLEY OGILVY, Hamilton College

The following short proof is offered for the main result of the paper by John H. Staib and Miltiades S. Demos in Vol. 40, page 210 of this MAGAZINE, namely that $\{\sin n\}$ is dense on $[-1, 1]$.

Let the points whose polar coordinates (r, θ) are $(1, 1)$, $(1, 2)$, $(1, 3)$, $\cdots$ be called integral points. Then the integral points are dense on the unit circle. For, assume that there exists an arc-length of magnitude $\epsilon > 0$ such that between θ and $\theta+\epsilon$ there are no integral points. Then there is another such gap whose initial point is at $\theta+1$, another whose initial point is at $\theta+2$, and so on. Two initial points can never coincide, or π would be rational. But an infinite number (indeed any finite number $> (2\pi)/\epsilon$) of such gaps must cover the circle, so that the circle would contain *no* integral points, an absurdity.

The set $\{\sin n\}$ is a projection of the integral points onto the interval $[-1, 1]$ of the Y-axis. This set is dense in $[-1, 1]$ because every open interval is the projection of two open arcs of the unit circle which, in turn, contain integral points.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before August 1, 1969.

PROPOSALS

719. *Proposed by J. A. H. Hunter, Toronto, Canada.*

This apt alphametic was suggested by D. Murdoch. Simple mathematics can be fun when taught properly! So what must *DROP OUTS* represent? .

$$\begin{array}{cccc} D & O & U & R \\ D & O & N & S \\ D & O & N' & T \\ S & T & O & P \\ D & R & O & P \\ \hline O & U & T & S \end{array}$$

720. *Proposed by Alfred Kohler, Long Island University.*

A device that shuffles cards always rearranges them in the same way relative to the order in which they are given to it. All of the hearts arranged in order from ace to king were put into the device, and then these shuffled cards were put into the device again to be shuffled a second time.

a. If the cards emerged in the order 10, 9, *Q*, 8, *K*, 3, 4, *A*, 5, *J*, 6, 2, 7, what order were the cards in after the first shuffle?
b. Show that it is impossible for the order of the cards after the second shuffle to be: *Q*, 8, 9, *J*, *K*, *A*, 4, 10, 5, 2, 7, 6, 3.
c. What is the minimum number of times that the device would have to shuffle thirteen cards before one could be certain that the cards had been restored at least once to their original order?

721. *Proposed by Charles W. Trigg, San Diego, California.*

The eye of an observer standing in a ditch is level with the surrounding horizontal terrain. At a distance of x inches he observes a vertical pole on which there are three marks that divide the pole from the bottom into segments a, b, c, and d. If the four segments subtend equal angles at the eye of the observer, then:

1. Find x;
2. Determine limiting conditions on a and b;
3. Ascertain the smallest integer values of x, a, b, c, and d;
4. Compute the height of the pole;
5. Calculate the angle subtended by each segment.

722. *Proposed by Gregory Wulczyn, Bucknell University, Pennsylvania.*

Prove that $n(n+1)(2n+1) \geqq \dfrac{12}{n-1} \sum_{i,j=1}^{n} ij;\ i<j.$

723. *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Find the ratio of the major axis to the minor axis of an ellipse which has the same area as its evolute.

724. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Find the probability that for a point P taken at random in the interior of a triangle ABC $(a \geqq b \geqq c)$, the distances of P from the sides of ABC form the lengths of sides of a triangle.

SOLUTIONS

Late Solutions

Dermott A. Breault, Harvard Computing Center: **691**; *Randolph Franklin, Lisgar Collegiate Institute, Ottawa, Canada*: **691, 692**; *Alexandru Lupas, Institutul de calcul, Cluj, Romania*: **692, 694.**

Ship, Captain and Crew

698. [September, 1968] *Proposed by Bertrand N. Bauer, Northwestern University.*

What is the probability of "qualifying" in the dice game "Ship-Captain-Crew"? The game is played with five dice. Each player, in his turn, throws all dice, or any portion of them he wishes, a maximum of three times. To qualify, a player must throw a 6 on one die (the "Ship"), a 5 on a second die (the "Captain") and a 4 on a third die (the "Crew"). He must throw them in that order, although getting them simultaneously is permitted.

Solution by Gerald J. Jerabek, Michigan Technological University.

Let $p(xy\bar{z})_r$ = probability of throwing at least one x, at least one y, and no z's on one toss of r dice. (See table on p. 97.)

Basic probabilities of 5 dice toss:

$p(\bar{6})_5 = (5/6)^5$, $p(5)_5 = 1 - (5/6)^5$, $p(5 \mid \bar{6})_5 = p(5)_5^* = 1 - (4/5)^5$, $p(5 \mid \bar{6})_{5*} = 1 - (3/4)^5$, $p(\bar{6})_5^* = (4/5)^5$.

Probabilities of 5 dice toss:

$p(65)_5 = p(5)_5 - [p(5 \mid \bar{6})_5][p(\bar{6})_5] = 2550/7776$, $p(65 \mid \bar{4})_5 = p(65)_5^* = p(5)_5^* - [p(5 \mid \bar{6})_5^*][p(\bar{6})_5^*] = 1320/3125$, $p(65\bar{4})_5 = [p(65 \mid \bar{4})_5][p(\bar{4})_5] = 1320/7776$, $p(654)_5 = p(65)_5 - p(65\bar{4})_5 = 1230/7776$, $p(6\bar{5})_5 = p(6)_5 - p(65)_5 = 2101/7776$, $p(\bar{6})_5 = 3125/7776$.

* Using "5 sided dice"

Possible ways of qualifying:

Winning Cases

Throws	1	2**	3	4	5	6	7	8	9
1st	654	$65\bar{4}$	$6\bar{5}$	$6\bar{5}$	$6\bar{5}$	$\bar{6}$	$\bar{6}$	$\bar{6}$	$\bar{6}$
2nd	—	4 / $\bar{4}$	54	$5\bar{4}$	$\bar{5}$	654	$65\bar{4}$	$6\bar{5}$	$\bar{6}$
3rd	—	— / 4	—	4	54	—	4	54	654

** Two cases combined for ease of calculations.

Probabilities of 4 dice toss:

$p(54)_4 = p(4)_4 - [p(4\mid\bar{5})_4][p(\bar{5})_4] = 302/1296$, $p(5\bar{4})_4 = p(5)_4 - p(54)_4 = 369/1296$, $p(\bar{5})_4 = 625/1296$.

Probabilities of 3 dice toss:

$p(4)_3 = 91/216$, $p(\bar{4})_3 = 125/216$, $p(4)_6 = 31031/46656$.

(Two consecutive throws of 3 dice.)

Case Probabilities

Case	$6^{15}\times$Probability
1	74,373,396,480
2	53,085,352,320
3	29,603,325,312
4	15,238,687,464
5	14,276,295,000
6	29,889,000,000
7	13,513,500,000
8	11,896,912,500
9	12,011,718,750

Sum = 253,888,187,826

(The probability of success for a given "case" is the product of the probabilities of each toss requirement for that "case".) Therefore, p(qualifying) $=253{,}888{,}187{,}826/6^{15}\approx .540$.

Note. If must yet be shown that the optimum strategy has been employed—that is, always keeping one "6", and one "5" (when allowable).

Check keeping one "6" when "6$\bar{5}$" has been thrown on first toss:

Let $w(3)=$probability of qualifying by Case 3.

Since $(w(3)+w(4)+w(5))/p(6\bar{5})>(w(6)+w(7)+w(8)+w(9))/p(\bar{6})$, the optimum strategy is to keep the "6".

In a similar manner it can be shown that, in each case, the optimum strategy consists of keeping one "6", and one "5" (when allowable).

Also solved by Jack C. Abad, San Francisco, California; John M. Howell, Los Angeles City College; Robert Otto, Central Michigan University and the proposer. One incorrect solution was received.

Heronic Triangles

699. [September, 1968] *Proposed by James G. Seiler, San Diego City College.*

Find an oblique Heronic triangle with sides of three nonzero digits each, such that the nine digits involved are distinct. A Heronic triangle is defined as one that has integers for the lengths of the sides and also an integer representing the area.

I. *Solution by the proposer.*

A Heronic triangle is defined as one that has integers for the lengths of the sides and also an integer representing the area. (See Dewey C. Duncan, *Arithmetic in a Liberal Education*, McGraw-Hill Book Company, 1967, prelim. ed.) A Heronic triangle may be formed using two Pythagorean triangles with a common leg which then becomes an altitude of the triangle formed; e.g., triangles with sides of 5, 12, 13 and 9, 12, 15 which produce a Heronic triangle with sides of 13, 14, 15 and an area of 84.

Solution to proposed problem: 387, 549, 612.

This solution was found to be a multiple of the primitive Heronic triangle with sides 43, 61, 68 formed by the two Pythagorean triangles with sides 11, 60, 61 and 32, 60, 68.

II. *Solution by E. P. Starke, Plainfield, New Jersey.*

Using Heronic triangles that happen to appear in the solution of Problem E 1982 (*American Mathematical Monthly*, September, 1968, p. 783) and triangles similar to them, we find easily one which has sides 432, 765, 819, with area 163296.

Also solved by Kenneth M. Wilke, Topeka, Kansas.

A Prime Sum

700. [September, 1968] *Proposed by Charles W. Trigg, San Diego, California.*

Find a prime number which is the sum of primes in two ways such that either set of addends and the sum together contain no duplicated digits.

The solution $5+19+23=5+13+29=47$ was obtained by *Richard Carr, Georgetown University; J. A. H. Hunter, Toronto, Canada; Richard A. Jacobson, Houghton College, New York; Delmar E. Searls, Houghton College, New York; E. P. Starke, Plainfield, New Jersey; and Zalman Usiskin, University of Michigan.*

The solution $389+7+5=367+29+5=401$ was obtained by *Leon Bankoff, Los Angeles, California. Don N. Page, William Jewell College, Missouri, obtained* $5+61+743=809=5+43+761$ *and the first solution above.*

W-digit Automorphs

701. [September, 1968] *Proposed by Gregory Wulczyn, Bucknell University.*

Martin Gardner defines a w-digit automorph, m, to be an integer whose square has the w digits of m at its tail end.

1. Prove that if m is a w-digit automorph with base r, then $(1\ 0 \cdots 0\ 1)_r - m_r$ is also a w-digit automorph where $1\ 0 \cdots 0\ 1$ contains $w-1$ zeros.

2. If $t>1$, $s_i \geqq 1$ how many automorphs does the base number $r_1^{s_1}=p_2^{s_2} \cdots p_t^{s_t}$ have?

Solution by the proposer.

(The $*$ denotes a block of $w-1$ zeros.)

(1) Let $m = (k_1k_2 \cdots k_w)_r$ and let $n = (h_1h_2 \cdots h_w)_r = (1\ 0 \cdot \overset{*}{\cdot} \cdot 0\ 1)_r - m.$

$$\text{Then } n^2 = \left|(1\ 0 \cdot \overset{*}{\cdot} \cdot 0\ 1)_r - m\right|^2 = (1\ 0 \cdot \overset{*}{\cdot} \cdot 0\ 1)_r \times (1\ 0 \cdot \overset{*}{\cdot} \cdot 0\ 1)_r$$

$$- (1\ 0 \cdot \overset{*}{\cdot} \cdot 0\ 1)_r \times m - m(1\ 0 \cdot \overset{*}{\cdot} \cdot 0\ 1)_r + (k_1 1_2 \cdots k_w)_r^2$$

$$= (1\ 0 \cdot \overset{*}{\cdot} \cdot 0\ 1)_r \times n - (\cdots k_1k_2 \cdots k_w)_r + (\cdots k_1k_2 \cdots k_w)_r$$

$$= (\cdots h_1h_2 \cdots h_w)_r \pm \underbrace{(\cdots 0\ 0 \cdots 0)_r}_{w \text{ zeros}}$$

$$= (\cdots h_1h_2 \cdots h_w)_r$$

so that n is also a w digit automorph.

(2) For each $p_i^{s_i}$ and $(\Pi\ p_i^{s_i})/(p_i^{s_i})$ there will be an automorph totalling $\binom{t}{1}+\binom{t}{t-1}$. For each $p_i^{s_i}\ p_j^{s_i}$ and $(\Pi\ p_i^{s_i})/(p_i^{s_i}p_j^{s_i})$ there will be an automorph totalling $\binom{t}{2}+\binom{t}{t-2} \cdots$

Also, 1 and $\Pi\ p_i^{s_i}$ are excluded as automorphs. An isomorphism is thus set up between the number of automorphs and the binomial expansion $(1+1)^t$, $t \geqq 2$ and the number of automorphs will be 2^t-2.

Note. There can be no automorph for p_i^a, $a<s_i$, since an automorph for p_i^a must be a multiple of p_i^a and its companion automorph must be a multiple of $(\Pi\ p_i^{s_i})/(p_i^a)$ or a multiple of p^{s_i-a} and the sum of the two automorphs must therefore be a multiple of p and equal $(11)_r$. But this is impossible since $((11)_r, p)=1$.

Also solved by Birger Jansson, Research Institute of National Defense, Stockholm, Sweden, who proved both parts of the problem as two theorems in a paper, On Genealogical Trees Containing Quadratic Residues, published in a Research Institute of National Defense report; and Kenneth M. Wilke, Topeka, Kansas.

Progression of Tangents

702. [September, 1968] *Proposed by R. Sivaramakrishnan, Government Engineering College, Trichur, India.*

If the line joining the circumcenter and orthocenter of a triangle ABC is parallel to a side, prove that $\tan A$, $\tan B$, and $\tan C$ are in arithmetic progression in some order.

Solution by John E. Homer, Jr., Union Carbide Corporation, Chicago.

In triangle ABC with circumcenter O and orthocenter H, assume OH is parallel to BC. Then $\tan B\cdot\tan C=3$ (see p. 18, H. S. M. Coxeter, *Introduction to Geometry*), and

$$\begin{aligned}\tan A &= \tan(180^\circ-(B+C))\\ &= \frac{\tan B+\tan C}{\tan B\cdot\tan C-1}\\ &= \frac{\tan B+\tan C}{2}.\end{aligned}$$

Therefore, $\tan A$, $\tan B$, and $\tan C$ are in A.P., with $\tan A$ the mean of $\tan B$ and $\tan C$.

Also solved by Jorges Andres, St. Francis College, New York; Leon Bankoff, Los Angeles, California; Mannis Charosh, Brooklyn, New York; Jack G. Deutsch, Erasmus Hall High School, Brooklyn, New York; Santo M. Diano, Philadelphia, Pennsylvania; Herta T. Freitag, Hollins College, Virginia; Michael Goldberg, Washington, D. C.; H. S. Hahn, West Georgia College; Don N. Page, William Jewel College, Missouri; Danny Strauss, James College, Stony Brook, New York; Charles W. Trigg, San Diego, California; Zalman Usiskin, University of Michigan; and the proposer.

Mannis Charosh showed the relation of the problem to E 259, *American Mathematical Monthly*, 1937, and Problem 1541, *School Science and Mathematics*, May, 1938.

A Divisible Polynomial

703. [September, 1968] *Proposed by A. H. Beiler, Brooklyn, New York, and J. A. H. Hunter, Toronto, Canada.*

Prove that for integral $n\geqq 1$, $(x+1)^n-x^n-1$ is divisible by x^2+x+1, if and only if $n=6m\pm1$.

Solution by L. Carlitz, Duke University.

Put $f(x)=(x+1)^n-x^n-1$, $x^2+x+1=(x-\omega)(x-\omega^2)$. Then it suffices to show that $f(\omega)=0$ if and only if $n=6m\pm1$.

Since $\omega^2+\omega+1=0$, it is clear that $f(\omega)=(-\omega^2)^n-\omega^n-1$. If $n\equiv 0 \pmod 3$, $f(\omega)=(-1)^n-2\neq 0$. If $n\equiv 2$ or $-2 \pmod 6$, then $f(\omega)=\omega^{2n}-\omega^n-1=-2(\omega^n+1)\neq 0$. Finally if $n\equiv 1$ or $-1 \pmod 6$, we have $f(\omega)=-\omega^{2n}-\omega^n-1=0$.

Remark. The result is not new; see Bachmann: *Das Fermatproblem in seiner bisherigen Entwicklung*, Berlin and Leipzig, 1919, p. 31. Moreover, when $n=6m+1$, $f(x)$ is divisible by $(x^2+x+1)^2$.

Also solved by Mannis Charosh, Brooklyn, New York; Michael Goldberg, Washington, D. C.; Jack G. Deutsch, Erasmus Hall High School, Brooklyn, New York; Randolph Franklin, Lisgar Collegiate Institute, Ottawa, Canada; Ernest F. Haeussler, Jr. and Richard S. Paul (jointly), Pennsylvania State University at Hazleton; H. S. Hahn, West Georgia College; Robert J. Herbold, Procter and Gamble Company, Cincinnati, Ohio; Lew Kowarski, Morgan State College, Maryland; Kenneth A. Ribet, Brown University; Ellis J. Rich, State University of New York Maritime College; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; Kenneth M. Wilke, Topeka, Kansas; Gregory Wulczyn, Bucknell University; and both proposers.

Adjoint Matrix Clues

704. [September, 1968] *Proposed by Maxey Brooke, Sweeny, Texas.*

The adjoint matrix of

$$\begin{bmatrix} I & A & -A \\ A & R & A \\ T & A & T \end{bmatrix}$$

is

$$\begin{bmatrix} X & -R & I \\ M & AM & -X \\ -X & -T & AA \end{bmatrix}$$

What digits are represented by each letter?

Solution by Zalman Usiskin, University of Michigan.

The definition of the adjoint matrix immediately gives $M=A\cdot T-A\cdot T$, $R=A(A+T)$, $I=A(A+R)$, and $X=A(A+I)$. The first of these equations gives $M=0$. Since R, I, and X are digits, the last three imply $A=1$. By this time, we have examined the six letters used in the problem, have psyched out Maxey Brooke, and guessed at the solution given below. All that remains is to mathematically demonstrate its uniqueness.

The definition of adjoint causes $AA=R\cdot I-A^2$, or $R\cdot I=12$. From the third equation given above, $I=1+R$. Thus $I=4$, $R=3$. From the other equations, $R=1+T$ and $X=1+I$. Thus $X=5$ and $T=2$. This gives the unique solution: $MATRIX=012345$.

Also solved by Gladwin E. Bartel, Washington State University; Donald Batman, MIT Lincoln Laboratory; Richard L. Breisch, University of Colorado; Martin J. Brown, University of Kentucky; Fred Ermis, Jr., Wharton County Junior College, Texas; Kathryn Ann Everson, Vassar College; Robert L. Hoburg, Western Connecticut State College; C. C. Oursler, Southern Illinois University at Edwardsville; John E. Prussing, University of California at San Diego; Delmar E. Searls, Houghton College, New York; W. J. Sonsin, Edgewood Arsenal, Maryland; E. P. Starke, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; Kenneth M. Wilke, Topeka, Kansas; Eugene J. Zirkel, Nassau Community College, New York; and the proposer. One incorrect solution was received.

Comment on Problem 644

644. [January and September, 1967] *Proposed by Harlan L. Umansky, Union City, New Jersey.*

Find all the rectangles in which the area and the perimeter equal the same integer. Do the same for right triangles, equilateral triangles, and squares.

Comment by Charles W. Trigg, San Diego, California.

The solution to this problem, as published in the September, 1967 issue, pages 224–225, dealt only with polygons with rational *sides*. This was too restrictive. The problem requires only that the *area* and *perimeter* be equal to the same integer.

Rectangles. Consider rectangles with sides $x+\sqrt{y}$ and $x-\sqrt{y}$. Then $P=4x=x^2-y=A$. Whereupon $y=x(x-4)$, $x\geqq 4$ and is rational with denominator of 1, 2, or 4. Thus there is an infinity of rectangles with equal integer areas and perimeters. This includes the three rectangles with rational sides given in the published solution.

Right Triangles. If the legs of a right triangle are $a=x+\sqrt{y}$ and $b=x-\sqrt{y}$, then $c=\sqrt{2(x^2+y)}$. Then $A=(x^2-y)/2=2x+\sqrt{2(x^2+y)}=P$. Clearing of radicals, $y^2-y(2x^2-8x+8)+(x^4-8x^3+8x^2)=0$, whereupon $y=x^2$, which gives a null b, or $y=x^2-8x+8$ to which correspond $c=2(x-2)$ and $A=P=4(x-1)$, $x>(13/2)$ with denominator of 1, 2, or 4.

Thus there is an infinity of right triangles with equal integer areas and perimeters. This includes the three triangles given in the previous solution.

Comment on Problem 691

691. [May, 1968, and January, 1969] *Proposed by Charles W. Trigg, San Diego, California.*

Using the nine positive digits just once each, form two integers A and B such that $A=8B$.

TABLE 1. $S=\{1, 2, \cdots, b-1\}$

b	m = 2	3	4	5	6	7	8	9	10	11	12	13	14	Row Total
3	1													1
4	1	1												2
5	0	0	0											0
6	0	0	2	1										3
7	5	0	1	0	0									6
8	3	3	3	2	15	1								27
9	2	24	2	5	1	0	0							34
10	12	2	4	12	3	7	46	3						89
11	36	0	31	0	4	0	0	0	0					81
12	64	17	15	9	64	9	30	5	171	13				397
13	516	0	52	137	10	0	20	9	2	0	0			746
14	0	32	48	91	79	0	36	63	64	49	866	49		1377
15	960	0	84	0	242	0	55	0	19	0	7	0	0	1367

Comment by David E. Daykin, University of Malaya, Kuala Lumpur, Malaysia. Please see Table 1 on page 102.

For each base $b \geqq 2$ the positive integers have unique representations of the form

(1) $$\alpha_k b^k + \cdots + \alpha_1 b^1 + \alpha_0 b^0, \text{ where } 0 \leqq \alpha_i < b \text{ for } 0 \leqq i \leqq k.$$

I wrote a computer program which, for each base b and multiplier m in the range $2 \leqq m < b \leqq 15$, found all solutions of the equation $A = mB$ such that, between the representations of the form (1) of A and B, each of the numbers in a given set S occurs exactly once as an α_i, and every α_i is in S. The numbers of solutions obtained appear in the tables.

TABLE 2. $S = \{0, 1, \cdots, b-1\}$

b	m: 2	3	4	5	6	7	8	9	10	11	12	13	14	Row Total
3	0													0
4	1	0												1
5	1	0	0											1
6	0	0	2	1										3
7	3	0	1	0	2									6
8	9	3	3	2	7	1								25
9	6	48	3	7	6	0	5							75
10	48	6	8	12	0	1	16	3						94
11	0	0	116	0	37	0	31	0	14					198
12	320	47	30	10	64	9	12	5	34	3				534
13	84	0	82	719	95	0	332	267	122	0	54			1755
14	0	96	158	154	69	0	26	50	15	11	161	10		750
15	0	0	428	0	1992	0	935	0	455	0	425	0	226	4461

TABLE 3

S	Total No. of Solutions
$\{i: 1 \leqq i < b,\ i \text{ odd}\}$	6
$\{i: 2 \leqq i < b,\ i \text{ even}\}$	14
$\{i: 0 \leqq i < b,\ i \text{ even}\}$	2

I found these results very surprising.

Comment on Problem 697

679. [January and September, 1968] *Proposed by Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania.*

Prove that $2^\alpha < 1 + \alpha$ for $0 < \alpha < 1$.

Comment by Robert G. Van Meter, St. Lawrence University, New York.

Solution **II** is faulty. A *reductio ad absurdum* proof should have started with the following supposition:

$$\sim (\forall x)(0 < x < 1 \rightarrow 2^x < 1 + x),$$

that is,

$$(\exists x)(0 < x < 1 \text{ and } 2^x \geqq 1 + x).$$

Thus the limit argument breaks down.

In addition, the "proof" given is a proof of the following statement:

$$\sim (\forall x)(0 < x < 1 \rightarrow 2^x \geqq 1 + x),$$

that is,

$$(\exists x)(0 < x < 1 \text{ and } 2^x < 1 + x)!$$

Similar comments were submitted by John D. Baum, Oberlin College, Ohio; Miltiades S. Demos, Villanova University, Pennsylvania; J. F. Leetch, Bowling Green State University, Ohio; Gary B. Weiss, New York University School of Medicine; and Donald W. Western, Franklin and Marshall College, Pennsylvania.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q449. Let $F(n)$ be the number of sequences of n symbols consisting of zeros and ones such that any zeros which appear must be separated into blocks whose lengths are odd. Prove that $F(n)=F(n-1)+2F(n-2)-F(n-3)$.

[*Submitted by Erwin Just*]

Q450. Let x be a rational number. If $\sqrt{24x^2+8x+1}$ is a rational number, then $x=k/2(k-1)^2-3$, where k is a rational number.

[*Submitted by Masaaki Shiba, Fukushima City, Japan*]

Q451. Let f be an everywhere differentiable function of one real variable. Show that f' has no removable discontinuities.

[*Submitted by Hugh Noland*]

Q452. Using as dimensions all of the first ten natural numbers, find five rectangles that can be assembled into a square.

[*Submitted by Benjamin Sharpe*]

Q453. Let F be a finite field of order $p+1$, where p is an odd prime. Prove that p is a Mersenne prime.

[*Submitted by Charles C. Lindner*]

(Answers on page 90)

By the mean value theorem for the definite integral we obtain

$$h = \frac{1}{\sqrt{1 - \cos^2 u}} (\cos x - \cos (x + h)),$$

where $\cos(x+h) \leqq \cos u \leqq \cos x$, since $x \leqq u \leqq (x+h)$ and the cosine function is monotone decreasing in the first quadrant. Then $-\sin u = -\sqrt{\sin^2 u} = (\cos(x+h) - \cos x)/h$ since $\sin u > 0$. As $h \to 0$, then $\sin(x+h) \to \sin x$ and $\sin u \to \sin x$ since $x \leqq u \leqq (x+h)$, so that

$$-\sin x = \lim_{h \to 0} \frac{\cos (x + h) - \cos x}{h} = D_x (\cos x).$$

For both Theorems 1 and 2 above, it is assumed that the sine and cosine functions are continuous functions.

The author wishes to thank the reviewer for his valuable suggestions.

ANSWERS

A449. The sequences consisting of n zeros and ones may be separated into sets such that each set contains sequences which begin with one of the following blocks: 1, 01, 0001, 000001, $\cdots$, or all zeros if n is odd. It follows that

$$\begin{aligned} F(n) &= F(n-1) + F(n-2) + F(n-4) + F(n-6) + \cdots \\ &= F(n-1) + F(n-2) - F(n-3) + F(n-3) + F(n-4) \\ &\quad + F(n-6) + \cdots \\ &= F(n-1) + 2F(n-2) - F(n-3). \end{aligned}$$

A450. If we put $z = -4 + 1/x$ where $x \neq 0$ we obtain the result from the fact that $\sqrt{z^2+8}$ is rational.

A451. f' has a removable discontinuity at t if and only if $\lim_{x\to t} f'(x)$ exists and is different from $f'(t)$. This never happens for $f'(t)$ is defined as $\lim_{x\to t}(f(x) - f(t))/(x-t)$ and we apply l'Hospital's rule in evaluating this limit, thus obtaining $f'(t) = \lim_{x\to t} f'(x)$.

A452. Proceeding clockwise: 8×2, 1×9, 10×5, 3×6 surround 4×7. This solution is not unique.

A453. Evidently $p+1$ is divisible by 2. Since every finite field has order q^n where q is prime and n is a positive integer, the order of F must be 2^n, that is, $p+1 = 2^n$. Hence $p = 2^n - 1$ is a Mersenne prime.

(Quickies on page 104)

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